Robust Transmit Beamforming for Parameter Estimation Using Distributed Sensors

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***Abstract*— In this letter, we study the problem of estimating a complex parameter in a wireless sensor network under individual power constraints and bounded channel uncertainty, where the channel state information is assumed to be imperfect at both the sensors and the fusion center. Transmit beamformers and a linear estimator that minimize the maximum mean square error are found. Although this problem is nonconvex, it can be relaxed to become a semidefinite programming problem by re-parameterization and semidefinite relaxation. Surprisingly, we find that relaxation is tight and a global optimal solution can be found. Finally, numerical simulations are performed to evaluate the performance of the robust estimator.**

***Index Terms*— Robust estimation, collaborative beamforming, SDR, SDP.**

I. Introduction

DISTRIBUTED parameter estimation in wireless sensor networks (WSNs) has drawn a great deal of attention in recent years. Due to resource limitations such as power and bandwidth constraints, one has to design the system carefully in order to achieve good performance. As we consider estimation of a continuous parameter, here we employ an analog communication scheme, which forwards the noisy sensor observations to the fusion center through a multiple access channel (MAC). Since the power of each sensor is limited, one needs to design the beamformers and the linear estimator to improve the reconstruction performance under individual sensor power constraints. In practical estimation problems, the system parameters may not be known precisely. In this setting, a popular approach is to use the minimax criterion to estimate the unknown parameters [1]. In [2], the optimal linear estimators are designed to cope with the worst-case model uncertainty. It is shown that the robust methods provide stable solutions compared to the classical least-squares solution. In [1] and [2], the estimation algorithm is optimized without considering the communications, although generally these optimizations are coupled.

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In this letter, the problem of finding the optimal beamformers and the linear estimation of a complex parameter is studied under both individual power constraints at each sensor and bounded channel uncertainty at both the sensors and fusion center.

One related problem is the robust linear transceiver design problem in MIMO communication systems, as shown in [3]–[7], [14]. In [3] and [4], the unknown channel is modeled as a random Gaussian matrix whose statistics are known, while in [5]–[7], the channel uncertainty is bounded like ours. By assuming that either the transmitters or the receivers are fixed, alternating optimization methods are utilized to find a stationary point, while, unlike in our work, a global optimum is not guaranteed. Compared to our work, the power constraint is different in that we consider individual power constraints while [5] and [6] consider a sum power constraint. We assume a more realistic assumption that the sensors produce noisy measurements, while [5]–[7] do not. As a consequence, the variance of the equivalent noise at the fusion center is related to the beamformers and the unknown channel in our letter, while the variance of the equivalent noise in [5] and [6] is constant, which results in a different problem formulation. Finally, we can prove that the optimal solutions can be found efficiently by standard SDP algorithms, which is a major contribution of this letter.

Some previous work [8] focused on robust collaborative beamforming without considering an estimation problem. In this setting, they propose semidefinite relaxation (SDR) plus bisection methods to find the beamformers, and the rank-one optimal solution is guaranteed under certain conditions. In our letter, we are able to relate our work to that in [8] in an interesting way if we assume that the receiver uses a linear estimator to reconstruct the parameter, and the worst-case MSE is utilized as a metric to jointly design the robust beamformer and estimator. Interestingly, we show that the difference lies in channel uncertainty models. If the channel state information (CSI) at the fusion center is perfect, then the resulting minimax MSE problem will coincide with the maximin SINR problem [8].

II. Preliminary

Suppose that there are N sensors deployed in the region of interest. The ith sensor will forward its noisy observation x_i which is related to the parameter $\theta \in \mathbb{C}$ via

$$x_i = \theta + v_i, \quad i = 1, \cdots, N, \tag{1}$$

where υ_i is circularly-symmetric complex Gaussian noise. It is known that the prior information about θ is $\theta \sim \mathcal{CN}(0, \sigma_\theta^2)$. Let $\boldsymbol{\upsilon}$ be $\boldsymbol{\upsilon} = [\upsilon_1, \cdots, \upsilon_N]^{\mathrm{T}}$ and assume that $\boldsymbol{\upsilon} \sim \mathcal{CN}(\mathbf{0}, \boldsymbol{\Sigma})$. Note that $\boldsymbol{\Sigma}$ need not be a diagonal matrix, as the noise of sensors may be spatially correlated. The parameter θ in (1) could represent the complex envelope representation of a narrow band signal [10], a common and powerful formulation. As a particular example of (1), suppose we have an array of closely spaced sensors that are about the same distance from an object omitting a narrow band sound wave at a known frequency. Assume the goal is to estimate the amplitude and phase of the narrow band signal. Maybe the sound is that made by a rotating tire and the amplitude and phase provide some information about its distance and orientation.

The ith sensor then forwards its noisy observation x_i to the fusion center by encoding the analog information into the amplitude and phase of a carrier signal. Before transmitting to the fusion center, we assume that the sensors will amplify and phase shift the transmitted signal as denoted by multiplication by a complex factor of α_i. Since the output power of sensors can not be arbitrary large, we require

$$\mathrm{E}|\alpha_i x_i|^2 = |\alpha_i|^2(\sigma_\theta^2 + \Sigma_{ii}) \leq p_i, \quad i = 1, \cdots, N. \tag{2}$$

A coherent multiple access scheme is used in this wireless sensor network. Define $\tilde{\mathbf{x}} = [\alpha_1 x_1, \cdots, \alpha_N x_N]^{\mathrm{T}}$ and $\mathbf{h} = [h_1, \cdots, h_N]^{\mathrm{T}}$, where h_i denotes the complex channel coefficient between the ith sensor and fusion center. Consequently, the baseband signal received by the fusion center is

$$y = \mathbf{h}^{\mathrm{T}}\tilde{\mathbf{x}} + n = \sum_{i=1}^{N} h_i \alpha_i \theta + (\sum_{i=1}^{N} h_i \alpha_i \upsilon_i + n), \tag{3}$$

where n is the circularly-symmetric Gaussian noise satisfying $n \sim \mathcal{CN}(0, \sigma_n^2)$. The second term of (3) is the noise term which is denoted by w. It follows by straightforward calculation that the variance of w is

$$\sigma_w^2 = \mathbf{h}^{\mathrm{T}}(\boldsymbol{\alpha}\boldsymbol{\alpha}^{\mathrm{H}} \circ \boldsymbol{\Sigma})\mathbf{h}^* + \sigma_n^2, \tag{4}$$

where $\mathbf{A} \circ \mathbf{B}$ denotes the Hadamard product of the two matrices and $\mathbf{h}^*$ denotes the conjugate of $\mathbf{h}$.

We first consider the case in which $\mathbf{h} = [h_1, h_2, \cdots, h_N]^{\mathrm{T}}$ is known precisely at both the sensors and fusion center. We design a linear estimator $\hat{\theta} = c^* y$ and beamforming scheme such that the mean square error (MSE) is minimized, where c^* denotes the conjugate of c. Note that the MSE is

$$\begin{aligned}&\mathrm{MSE}(\hat{\theta}, \theta)\\ &= \mathrm{E}|\hat{\theta} - \theta|^2 \triangleq f(\boldsymbol{\alpha}, c, \mathbf{h})\\ &= |\mathbf{h}^{\mathrm{T}}(c^*\boldsymbol{\alpha}) - 1|^2\sigma_\theta^2 + \mathbf{h}^{\mathrm{T}}\left((c^*\boldsymbol{\alpha})(c^*\boldsymbol{\alpha})^{\mathrm{H}} \circ \boldsymbol{\Sigma}\right)\mathbf{h}^* + |c|^2\sigma_n^2.\end{aligned} \tag{5}$$

Consequently, finding both the optimal beamform vector and linear estimator amounts to

$$\underset{c, \boldsymbol{\alpha} \in \mathcal{A}}{\text{minimize}} \quad \mathrm{MSE}(\hat{\theta}, \theta), \tag{6}$$

where $\mathcal{A}$ is defined as

$$\mathcal{A} = \{\boldsymbol{\alpha} | |\alpha_i|^2(\sigma_\theta^2 + \Sigma_{ii}) \leq p_i, i = 1, \cdots, N.\}. \tag{7}$$

Note that problem (6) is nonconvex due to the bilinear term $c^*\boldsymbol{\alpha}$. However, we will show that this problem can be solved efficiently by re-parameterization. By introducing a new variable $\boldsymbol{\beta} = c^*\boldsymbol{\alpha}$ and $\mathbf{h}^{\mathrm{T}}(\boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{H}} \circ \boldsymbol{\Sigma})\mathbf{h}^* = \boldsymbol{\beta}^{\mathrm{H}}(\mathbf{h}^*\mathbf{h}^{\mathrm{T}} \circ \boldsymbol{\Sigma}^*)\boldsymbol{\beta}$, we can obtain another optimization problem

$$\underset{c, \boldsymbol{\beta}}{\text{minimize}} \ g(\boldsymbol{\beta}, c, \mathbf{h}) \tag{8a}$$

$$\text{subject to } |\beta_i|^2(\sigma_\theta^2 + \Sigma_{ii}) \leq p_i|c|^2, \quad i = 1, \cdots, N, \tag{8b}$$

where

$$\begin{aligned}g(\boldsymbol{\beta}, c, \mathbf{h}) &\triangleq \boldsymbol{\beta}^{\mathrm{H}}\mathbf{M}_{\mathrm{eq}}\boldsymbol{\beta} - 2\mathrm{Re}(\mathbf{h}^{\mathrm{T}}\boldsymbol{\beta})\sigma_\theta^2 + |c|^2\sigma_n^2 + \sigma_\theta^2,\\ &= \mathbf{h}^{\mathrm{H}}\left(\boldsymbol{\beta}^*\boldsymbol{\beta}^{\mathrm{T}} \circ (\sigma_\theta^2\mathbf{1}\mathbf{1}^{\mathrm{T}} + \boldsymbol{\Sigma}^*)\right)\mathbf{h} - 2\mathrm{Re}(\mathbf{h}^{\mathrm{H}}\boldsymbol{\beta}^*)\sigma_\theta^2 + |c|^2\sigma_n^2 + \sigma_\theta^2\end{aligned} \tag{9}$$

and $\mathbf{M}_{\mathrm{eq}} = \mathbf{h}^*\mathbf{h}^{\mathrm{T}} \circ (\boldsymbol{\Sigma}^* + \sigma_\theta^2\mathbf{1}\mathbf{1}^{\mathrm{T}})$. For optimization problems (6) and (8), we have the following results.

Proposition 1: Problem (6) is equivalent to (8).

Proof: Basically, (6) becomes (8a) and (7) becomes (8b). ■

Interestingly, problem (8) is only related to the norm of c. Thus we introduce $e = |c|^2$. Then the optimization variables are $(\boldsymbol{\beta}, e)$, where the objective function and constraints are both linear in the variable e. Note that the objective function of the above optimization problem is convex, and the constraint is a cone, thus this problem can be solved efficiently by using the convex optimization toolbox *CVX*.

Next we study the case in which the channels are uncertain at both the sensors and the fusion center. This may occur if the training time is limited, then the channel can not be estimated precisely. We use a bounded uncertainty model and assume that the channel $\mathbf{h}$ lies in the following uncertainty set

$$\mathcal{H} = \{\mathbf{h} | \|\mathbf{h} - \mathbf{h}_0\|_{\mathbf{T},2} \leq \rho_h\}, \tag{10}$$

where $\mathbf{h}_0$ is the nominal channel, $\|\mathbf{x}\|_{\mathbf{T},2} = \sqrt{\mathbf{x}^{\mathrm{H}}\mathbf{T}\mathbf{x}}$, $\mathbf{T}$ is a positive definite matrix, and ρ_h is a known constant which characterizes the uncertainty level of $\mathbf{h}$. Since the minimax criterion has been widely adopted in the robust design framework [1], the robust beamformer and estimator are designed via

$$(c_{\mathrm{opt}}, \boldsymbol{\alpha}_{\mathrm{opt}}) = \underset{c, \boldsymbol{\alpha} \in \mathcal{A}}{\operatorname{argmin}} \max_{\mathbf{h} \in \mathcal{H}} \mathrm{MSE}(\hat{\theta}, \theta), \tag{11}$$

where $\mathcal{A}$ is defined in (7).

III. Robust Design of Beamformer and Estimator

In this section, we show that the robust design of beamformers and estimators can be jointly found. For the robust optimization problem (11), by introducing a new variable t and the same new variable $\boldsymbol{\beta}$ as shown in (8), problem (11) can be reformulated as

$$\underset{t, c, \boldsymbol{\beta}}{\text{minimize}} \ t \tag{12a}$$

$$\text{subject to } g(\boldsymbol{\beta}, c, \mathbf{h}) \leq t, \forall \ \mathbf{h} \in \mathcal{H} \tag{12b}$$

$$|\beta_i|^2(\sigma_\theta^2 + \Sigma_{ii}) \leq p_i|c|^2, \quad i = 1, \cdots, N. \tag{12c}$$

Based on the S procedure [11], the infinite constraints (12b) are equivalent to that there exists some $s \geq 0$ such that (13) holds, where (13) is at the top of the next page and

$$\mathbf{B} = \boldsymbol{\beta}\boldsymbol{\beta}^{\mathrm{H}}. \tag{14}$$

$$\begin{bmatrix} s\mathbf{T} - \left(\mathbf{B}^* \circ (\sigma_\theta^2 \mathbf{1}\mathbf{1}^{\mathrm{T}} + \mathbf{\Sigma}^*)\right) & \boldsymbol{\beta}^*\sigma_\theta^2 - s\mathbf{T}\mathbf{h}_0 \\ (\boldsymbol{\beta}^*\sigma_\theta^2 - s\mathbf{T}\mathbf{h}_0)^{\mathrm{H}} & t - (\sigma_\theta^2 + |c|^2\sigma_n^2) - s(\rho_h^2 - \|\mathbf{h}_0\|_{\mathbf{T},2}^2) \end{bmatrix} \succeq \mathbf{0} \tag{13}$$

$$s\mathbf{T} - \left(\mathbf{B}^* \circ (\sigma_\theta^2 \mathbf{1}\mathbf{1}^{\mathrm{T}} + \mathbf{\Sigma}^*)\right) - (\boldsymbol{\beta}^*\sigma_\theta^2 - s\mathbf{T}\mathbf{h}_0)(\boldsymbol{\beta}^*\sigma_\theta^2 - s\mathbf{T}\mathbf{h}_0)^{\mathrm{H}}/(t - (\sigma_\theta^2 + |c|^2\sigma_n^2) - s(\rho_h^2 - \|\mathbf{h}_0\|_{\mathbf{T},2}^2)) \succeq \mathbf{0} \tag{17}$$

The above constraint (14) is nonconvex and can be relaxed as

$$\begin{bmatrix} \mathbf{B} & \boldsymbol{\beta} \\ \boldsymbol{\beta}^{\mathrm{H}} & 1 \end{bmatrix} \succeq \mathbf{0}. \tag{15}$$

As a consequence, we can use *CVX* to solve the following SDP problem

$$\underset{t,c,s,\mathbf{B},\boldsymbol{\beta}}{\text{minimize}}\ t \tag{16a}$$

$$\text{subject to (13), (15) and (12}c). \tag{16b}$$

Meanwhile, we obtain the following results.

Proposition 2: Given that $(t_{\mathrm{opt}}, c_{\mathrm{opt}}, s_{\mathrm{opt}}, \mathbf{B}_{\mathrm{opt}}, \boldsymbol{\beta}_{\mathrm{opt}})$ *are optimal solutions of (16),* $(t_{\mathrm{opt}}, c_{\mathrm{opt}}, s_{\mathrm{opt}}, \mathbf{B}_0 = \boldsymbol{\beta}_{\mathrm{opt}}\boldsymbol{\beta}_{\mathrm{opt}}^{\mathrm{H}}, \boldsymbol{\beta}_{\mathrm{opt}})$ *are also optimal for the original problem, where the original problem is (16) except that constraint (15) in (16b) is replaced with (14).*

Proof: Suppose that the optimal objective value of the original problem is $\tilde{t}_{\mathrm{opt}}$. We have $t_{\mathrm{opt}} \leq \tilde{t}_{\mathrm{opt}}$, because the feasible set of the relaxed problem (16) is larger. We will show that if we replace $\mathbf{B}_{\mathrm{opt}}$ with $\boldsymbol{\beta}_{\mathrm{opt}}\boldsymbol{\beta}_{\mathrm{opt}}^{\mathrm{H}}$, then $(t_{\mathrm{opt}}, c_{\mathrm{opt}}, s_{\mathrm{opt}}, \mathbf{B}_0 = \boldsymbol{\beta}_{\mathrm{opt}}\boldsymbol{\beta}_{\mathrm{opt}}^{\mathrm{H}}, \boldsymbol{\beta}_{\mathrm{opt}})$ is also a feasible point of the relaxed problem (16). Of course, it is also a feasible point of the original problem. Because the objective value evaluated at a feasible point must be greater than or equal to the global minimum, we have $t_{\mathrm{opt}} \geq \tilde{t}_{\mathrm{opt}}$. As a consequence $t_{\mathrm{opt}} = \tilde{t}_{\mathrm{opt}}$.

Now we prove that $(t_{\mathrm{opt}}, c_{\mathrm{opt}}, s_{\mathrm{opt}}, \mathbf{B}_0 = \boldsymbol{\beta}_{\mathrm{opt}}\boldsymbol{\beta}_{\mathrm{opt}}^{\mathrm{H}}, \boldsymbol{\beta}_{\mathrm{opt}})$ is a feasible point of the relaxed problem (16). We only need to investigate constraint (13). If $t - (\sigma_\theta^2 + |c|^2\sigma_n^2) - s(\rho_h^2 - \|\mathbf{h}_0\|_{\mathbf{T},,2}^2) > 0$, then constraint (13) is equivalent to (17), as shown at the top of this page, by Schur's complement [12]. Note that constraint (15) implies that $\mathbf{B}_{\mathrm{opt}} \succeq \mathbf{B}_0 = \boldsymbol{\beta}_{\mathrm{opt}}\boldsymbol{\beta}_{\mathrm{opt}}^{\mathrm{H}}$. According to the Schur product theorem [12], we have

$$(\mathbf{B}_{\mathrm{opt}}^* - \mathbf{B}_0^*) \circ (\sigma_\theta^2 \mathbf{1}\mathbf{1}^{\mathrm{T}} + \mathbf{\Sigma}^*) \succeq \mathbf{0}. \tag{18}$$

Because $(t_{\mathrm{opt}}, c_{\mathrm{opt}}, s_{\mathrm{opt}}, \mathbf{B}_{\mathrm{opt}}, \boldsymbol{\beta}_{\mathrm{opt}})$ satisfies constraint (17), the solution $(t_{\mathrm{opt}}, c_{\mathrm{opt}}, s_{\mathrm{opt}}, \mathbf{B}_0 = \boldsymbol{\beta}_{\mathrm{opt}}\boldsymbol{\beta}_{\mathrm{opt}}^{\mathrm{H}}, \boldsymbol{\beta}_{\mathrm{opt}})$ also satisfies constraint (17) according to (18) and is a feasible point of the relaxed problem (16). The case in which $t - (\sigma_\theta^2 + |c|^2\sigma_n^2) - s(\rho_h^2 - \|\mathbf{h}_0\|_{\mathbf{T},2}^2) = 0$ can be proved similarly. ■

Note that the developed algorithm is a centralized algorithm. Once the fusion center solves the SDP, it then transmits the coefficient α_i to the ith sensor. Because problem (16) has 3 constraints and matrices of size at most $N+1$ by $N+1$, the computational cost of the proposed algorithm is $O(N^3)$.

IV. Relationship Between Minimax MSE and Maximin SINR

In [8], the maximin signal-to-interference-plus-noise ratio (SINR) criterion is adopted for designing the beamformers, but [8] does not propose an estimation approach. As we show next, the maximin signal-to-interference-plus-noise ratio (SINR) criterion can also be viewed as a minimax MSE problem with perfect channel knowledge at the fusion center.

Note that if the fusion center knows the channel precisely [9], then it uses an optimal linear estimator and the resulting robust MSE optimization problem is

$$\min_{\boldsymbol{\alpha}\in\mathcal{A}} \max_{\mathbf{h}\in\mathcal{H}} \min_{c} \mathrm{MSE}(\hat{\theta}, \theta). \tag{19}$$

Let $\mathrm{MSE}_{\mathrm{opt}}$ and $\widetilde{\mathrm{MSE}}_{\mathrm{opt}}$ denote the optimal value of (11) and (19). According to [7], the inequality $\min_x \max_y f(x,y) \geq \max_y \min_x f(x,y)$ holds. We may define the gap and have

$$\gamma_{\mathrm{gap}} \triangleq \mathrm{MSE}_{\mathrm{opt}} - \widetilde{\mathrm{MSE}}_{\mathrm{opt}} \geq 0. \tag{20}$$

For problem (19), we eliminate variable c by setting the first derivative of $\mathrm{MSE}(\hat{\theta}, \theta)$ with respect to c^* to zero and obtain

$$c_{\mathrm{opt}} = \frac{\mathbf{h}^{\mathrm{T}}\boldsymbol{\alpha}\sigma_\theta^2}{\mathbf{h}^{\mathrm{T}}(\boldsymbol{\alpha}\boldsymbol{\alpha}^{\mathrm{H}} \circ \mathbf{\Sigma})\mathbf{h}^* + \sigma_n^2 + |\mathbf{h}^{\mathrm{T}}\boldsymbol{\alpha}|^2\sigma_\theta^2}, \tag{21}$$

Substituting $c = c_{\mathrm{opt}}$ in (19) produces

$$\min_{\boldsymbol{\alpha}\in\mathcal{A}} \max_{\mathbf{h}\in\mathcal{H}} \sigma_\theta^2/(\mathrm{SINR}(\mathbf{h}, \boldsymbol{\alpha}) + 1), \tag{22}$$

where $\mathrm{SINR}(\mathbf{h}, \boldsymbol{\alpha})$ is

$$\mathrm{SINR}(\mathbf{h}, \boldsymbol{\alpha}) = |\mathbf{h}^{\mathrm{T}}\boldsymbol{\alpha}|^2\sigma_\theta^2/(\mathbf{h}^{\mathrm{T}}(\boldsymbol{\alpha}\boldsymbol{\alpha}^{\mathrm{H}} \circ \mathbf{\Sigma})\mathbf{h}^* + \sigma_n^2), \tag{23}$$

which is the beamforming SINR according to (3) and (4), see [8]. Furthermore, problem (22) can be shown to be equivalent to the problem studied in [8]

$$\max_{\boldsymbol{\alpha}\in\mathcal{A}} \min_{\mathbf{h}\in\mathcal{H}} \mathrm{SINR}(\mathbf{h}, \boldsymbol{\alpha}). \tag{24}$$

It is worthwhile noting that the nonzero gap will imply that (11) is not equivalent to (24). The gap γ_{gap} is due to the difference of the channel uncertainty models and depends on constraint (10). If there is no uncertainty in $\mathbf{h}$, i.e., the sensors and fusion center have perfect channel state information, then the gap γ_{gap} is zero and the MSE minimization problem (11) is equivalent to SINR maximization problem (24). Otherwise, the maximin SINR problem can be viewed as a relaxation of the minimax MSE problem. Intuitively, the gap γ_{gap} increases as the uncertainty of $\mathbf{h}$ increases, as (11) and (19) reveal.

V. Simulation Results

In this section, two numerical investigations are conducted to corroborate the theoretical results. The parameters are set as follows: $N = 4$, $\mathbf{T} = \mathbf{I}$, $\sigma_\theta^2 = 1$, $p_i = 5, i = 1, \cdots, N$. The variance of the additive noise is $\sigma_n^2 = 0.1$ at the receiver. For the noise covariance matrix $\mathbf{\Sigma}$ at the sensors, we assume that the noise at sensors are spatially correlated

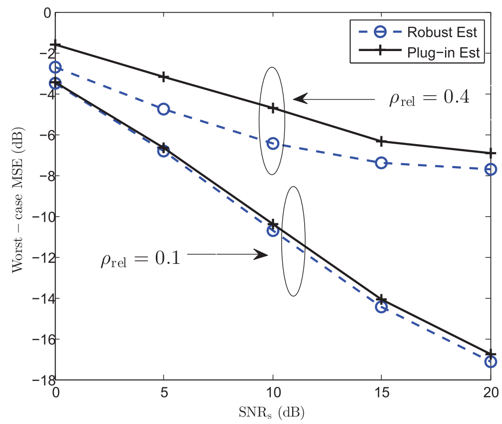

Fig. 1. The worst-case MSE of the various estimators versus the individual sensor SNR SNR_s, where the relative uncertainty levels ρ_{rel} are $\rho_{\text{rel}} = 0.1$ and $\rho_{\text{rel}} = 0.4$, respectively.

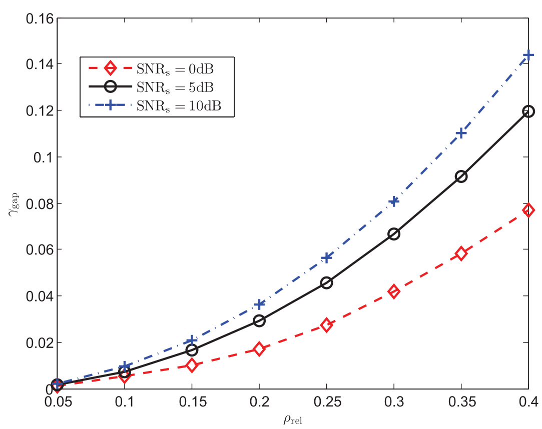

Fig. 2. The gap γ_{gap} (20) versus the relative uncertainty level ρ_{rel}.

and the covariance matrix $\mathbf{\Sigma}$ satisfies $\Sigma_{mn} = \sigma_v^2 * 0.8^{|m-n|}$. At each individual sensor it is assumed $\text{SNR}_s = 10 \log \frac{\sigma_\theta^2}{\sigma_v^2}$. The nominal channel $\mathbf{h}_0$ is generated from a Gaussian distribution $\mathcal{CN}(\mathbf{0}, \mathbf{I})$. We define $\rho_h = \rho_{\text{rel}} \|\mathbf{h}_0\|_2$, where ρ_{rel} can be viewed as the uncertainty level relative to the nominal channel norm. We first find the optimal plug-in beamformers $\boldsymbol{\alpha}_{\text{plug}}$ and estimator c_{plug} assuming that the true channel is the nominal channel $\mathbf{h}_0$. Then we find the worst-case channel $\mathbf{h}_{\text{wor}}$ by solving the inner optimization problem in (11) with fixed $\boldsymbol{\alpha}_{\text{plug}}$ and c_{plug}, which can be solved efficiently as shown in [13]. We then evaluate the MSE for both the robust estimator and the plug-in estimator with the channel being $\mathbf{h}_{\text{wor}}$. The simulation setup is similar to [5] and [6]. The performance is averaged over the nominal channel $\mathbf{h}_0$ for 100 times.

For the first simulation, we compare the worst-case MSE performance of the various estimators versus the common individual sensor SNR_s. The relative perturbation levels are $\rho_{\text{rel}} = 0.1$ and $\rho_{\text{rel}} = 0.4$, and the results are plotted in Fig. 1. From Fig. 1, it can be seen that robust estimator works better than the plug-in estimator. Besides, the performance gain provided by the robust estimator increases as the uncertainty level becomes large.

The second simulation corroborates the results in Section IV, i.e., maximizing the worst-case SINR can be regarded as a relaxation of the robust MSE minimization problem (11). The parameters are the same as the previous simulation. From Fig. 2, it is demonstrated that γ_{gap} in (20) is nonzero when the channel is uncertain. Meanwhile, γ_{gap} is monotone increasing as the relative uncertainty level ρ_{rel} increases. For the fixed uncertainty level, the gap decreases as SNR_s at the individual sensors become lower. This agrees with the observation that, in the low SNR region the channel knowledge at the fusion center becomes less important.

VI. Conclusion

In this letter, we have studied the joint beamformer and linear estimator design problem under individual power constraints and bounded channel uncertainty for estimating a complex parameter. It was shown that the non-convex optimization problem can be relaxed as an SDP, which can be solved efficiently by interior point algorithms. In addition, it was proved that this relaxation is tight in finding the optimal beamformers. Furthermore, it was shown that the difference between our problem and the maximin SINR problem is due to channel uncertainty models. Finally, numerical investigations reveal the advantages of the robust beamformers compared to the plug-in estimator.

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