CONVERGENCE ANALYSIS ON A FAST ITERATIVE PHASE RETRIEVAL ALGORITHM WITHOUT INDEPENDENCE ASSUMPTION

Gen Li, Yuchen Jiao, and Yuantao Gu
Department of Electronic Engineering, Tsinghua University, Beijing, China
Email: \{g-li16, jiao-14\}@mails.tsinghua.edu.cn, gyt@tsinghua.edu.cn

**Background**

- Phase Retrieval problem
  - Recover a vector from magnitude measurements \( y_r = (|x_r|^2) / r = 1, 2, \ldots, m \)
  - Non-convex quadratic programs
- Algorithms
  - Classical
    - Error Reduction (ER) and its variants
  - Recently developed
    - PhaseLift [Candes’ 13]
    - Wirtinger Flow algorithm [Candes’ 15]
    - Kaczmarz algorithm [Wei’s 15]
  - Row action method, low computation complexity

**Main Result**

- Convergence result for KA in real case:
  - \( x^* \in \mathbb{R}^n, e_t = x_t - x^* \)
  - \( \alpha := \frac{m}{n} \)
  - With probability \( 1 - e^{-\Omega} \) where \( c \) is a constant depending on \( \epsilon \)
  - \( \mathbb{E} \left[ \left| e_t \right|^2 \right] - \mathbb{E} \left[ \left| e_{t-1} \right|^2 \right] \leq - \left( 1 - \frac{p}{\alpha} \right) \left( 1 - \frac{1}{\sqrt{\alpha - p}} \right) - \frac{2p \ln \alpha - e \vert \alpha - p \vert}{\alpha} + \frac{3p}{\alpha} \left( 1 + \frac{1}{\sqrt{\alpha - p}} + \frac{2\ln \alpha}{\alpha} \right) \epsilon \)
  - \( p \in (0, \alpha) \) is a parameter determined by \( |e_{t-1}| \)
  - Linear rate of convergence still holds without IA

**Sketch of Proof**

- Take expectation with respect to \( r \), instead of \( |e_{t-1}| \)
- Based on random matrix theory, find uniform bounds for arbitrary \( |e_{t-1}| \)

- \( e_t = (1 - \frac{1}{\ln n}) e_{t-1} + e_{t-2} \) \( \sum r \left( \arg \left( a_r^T x^* x_r \right) \right) + 4 \left( e_{t-2}^* x^* \right)^2 \) \( ||e_t|| \leq ||e_{t-1}||^2 - \frac{1}{m} \ leq \sum \left( \frac{u^* e_{t-1}}{|a_r|^2} \right)^2 + \frac{3}{m} \ leq \sum \left( \frac{u^* e_{t-1}}{|a_r|^2} \right)^2 \)
- Using probability inequalities to get the uniform bounds for the latter two items
- Reference for details

**Preliminary**

- Phase Retrieval via Randomized Kaczmarz algorithm (KA) for real case:
  - Input: \( (a_r, y_r), r = 1, \ldots, m \), initialization \( x_0 \) using the spectral method, \( t = 1 \)
  - Ensure: \( x_r \) as an estimate for \( x^* \)
    - while \( t \leq T \) do
    1. choose \( r \) randomly from \( \{1, \ldots, m\} \) uniformly
    2. \( x_t \leftarrow x_{t-1} + \frac{y_r a_r^T (a_r^T x_{t-1}) - y_r a_r^T x^*}{|a_r|^2} \)
    - end while
  - Simulation result: linear rate of convergence

- Independence assumption (IA)
  - Iterative variable \( x_t \) independent with measurements \( \langle a_r, y_r \rangle \)

**Random Matrix Theory**

- A \( \in \mathbb{R}^{n \times 2} \) entries \( \sim N(0,1/n) \) iid
  - With probability at least \( 1 - 2e^{-t^2/2} \)

- \( \Phi \in \mathbb{R}^{n \times 2} \) entries \( \sim N(0,1/n) \) iid
  - With probability at least \( 1 - 2e^{-t^2/2} \)

**Our previous analysis**

  - Not rigorous: using IA

**Random Matrix Theory**

- A \( \in \mathbb{R}^{n \times 2} \) entries \( \sim N(0,1/n) \) iid
  - With probability at least \( 1 - 2e^{-t^2/2} \)

- \( \Phi \in \mathbb{R}^{n \times 2} \) entries \( \sim N(0,1/n) \) iid
  - With probability at least \( 1 - 2e^{-t^2/2} \)