CONVERGENCE ANALYSIS ON A FAST ITERATIVE PHASE RETRIEVAL ALGORITHM WITHOUT INDEPENDENCE ASSUMPTION

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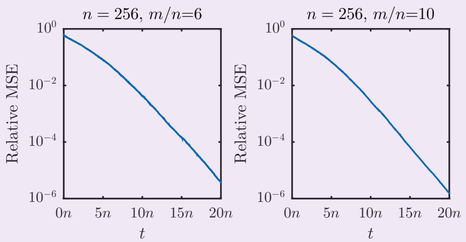
1 Background

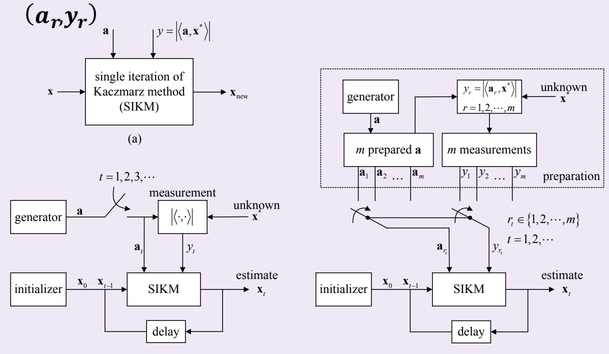
- Phase Retrieval problem
  - Recover a vector from magnitude measurements
    $y_r = |\langle a_r, x^*\rangle|, r = 1, 2, \dots, m$
  - Non-convex quadratic programs
- Applications: X-ray crystallography, Microscopy, Optics, Acoustics, Blind channel estimation.
- Algorithms
  - Classical
    - Error Reduction (ER) and its variants
  - Recently developed
    - PhaseLift [Candes' 13]
    - Wirtinger Flow algorithm [Candes' 15]
    - Kaczmarz algorithm [Wei's 15]
      - Row action method, low computation complexity

2 Preliminary

- Phase Retrieval via Randomized Kaczmarz algorithm (KA) for real case:
  - Input: $\{(a_r, y_r), r = 1, \dots, m\}$, initialization x_0 using the spectral method, $t = 1$.
  - Ensure: x_T as an estimate for x^*
    - while $t \le T$ do
    - 1. choose r randomly from $\{1, \dots, m\}$ uniformly
    - 2. $x_t \leftarrow x_{t-1} + \frac{y_r \mathrm{sgn}(a_r^T x_{t-1}) - a_r^T x_{t-1}}{\|a_r\|^2} a_r$ (1)
    - 3. $t \leftarrow t + 1$
    - end while
- Simulation result: linear rate of convergence



- Independence assumption (IA)
  - Iterative variable x_t independent with measurements (a_r, y_r)



  - (b) Infinite measurements, IA holds
  - (c) Finite measurements, IA wrong
- Our previous analysis
  - G. Li, Y. Gu, and Y. M. Lu, Phase Retrieval Using Iterative Projections: Dynamics in the Large Systems Limit.
  - Not rigorous: using IA

3 Random Matrix Theory

- $A \in \mathbb{R}^{n\times k}$: entries $\sim \mathcal{N}(0, 1/n)$ iid
- With probability at least $1 - 2e^{-nt^2/2}$

$$1 - \sqrt{\frac{k}{n}} - t \le \min_{\mathbf{v}} \|\mathbf{A}\mathbf{v}\| \le \max_{\mathbf{v}} \|\mathbf{A}\mathbf{v}\| \le 1 + \sqrt{\frac{kE}{n}} + t$$

- $\Phi \in \mathbb{R}^{n\times N}$: entries $\sim \mathcal{N}(0, 1/n)$ iid
- With probability at least $1 - 2\binom{N}{k}e^{-nt^2/2}$

$$1 - \sqrt{\frac{k}{n}} - t \le \min_{\|\mathbf{v}\|_0 \le k} \|\Phi\mathbf{v}\| \le \max_{\|\mathbf{v}\|_0 \le k} \|\Phi\mathbf{v}\| \le 1 + \sqrt{\frac{kE}{n}} + t$$

4 Main Result

- Convergence result for KA in real case:
  - $x^* \in \mathbb{R}^n$, $e_t = x_t - x^*$
  - $\alpha := \frac{m}{n}$
  - With probability $1 - e^{-cn}$ where c is a positive constant depending on ϵ

$$\frac{\mathbb{E}\|e_t\|^2 - \|e_{t-1}\|^2}{\|e_{t-1}\|^2/n} \le -\left(1 - \frac{p}{\alpha}\right)\left(1 - \frac{1}{\sqrt{\alpha - p}} - \sqrt{\frac{2p}{\alpha - p}\ln\frac{e\alpha}{p}}\right) + \frac{3p}{\alpha}\left(1 + \frac{1}{\sqrt{p}} + \sqrt{2\ln\frac{e\alpha}{p}}\right) + \epsilon,$$

  - $p \in (0, \alpha)$ is a parameter determined by $\|e_{t-1}\|$
  - Linear rate of convergence still holds without IA

5 Sketch of Proof

- Abandon independence assumption
  - Take expectation with respect to r, instead of $\|e_{t-1}\|$
  - Based on random matrix theory, find uniform bounds for arbitrary $\|e_{t-1}\|$
- $e_t = \left(I - \frac{a_r a_r^T}{\|a_r\|^2}\right)e_{t-1} + \frac{a_r^T x^*}{\|a_r\|^2} a_r \left(\mathrm{sgn}(a_r^T x^* a_r^T x_{t-1}) - 1\right)$
- $\|e_t\|^2 = \|e_{t-1}\|^2 - \left(\frac{a_r^T e_{t-1}}{\|a_r\|}\right)^2 + 4\left(\frac{a_r^T x^*}{\|a_r\|}\right)^2 \mathbb{I}(r \in \mathcal{S})$
- $\mathcal{S} := \{\mathrm{sgn}(a_r^T x_{t-1}) \ne \mathrm{sgn}(a_r^T x^*)\}$
- $\mathbb{E}\|e_t\|^2 = \|e_{t-1}\|^2 - \frac{1}{m}\sum_{k\in\bar{\mathcal{S}}}\left(\frac{a_k^T e_{t-1}}{\|a_k\|}\right)^2 + \frac{3}{m}\sum_{k\in\mathcal{S}}\left(\frac{a_k^T x^*}{\|a_k\|}\right)^2$
- Using probability inequalities to get the uniform bounds for the latter two items
- Reference for details
  - G. Li, Y. Jiao, and Y. Gu, "Linear convergence of an iterative phase retrieval algorithm with data reuse," 2017, arXiv:1712.01712 [cs.IT, cs.LG].