A JOINT DETECTION AND RECONSTRUCTION METHOD FOR BLIND GRAPH SIGNAL RECOVERY

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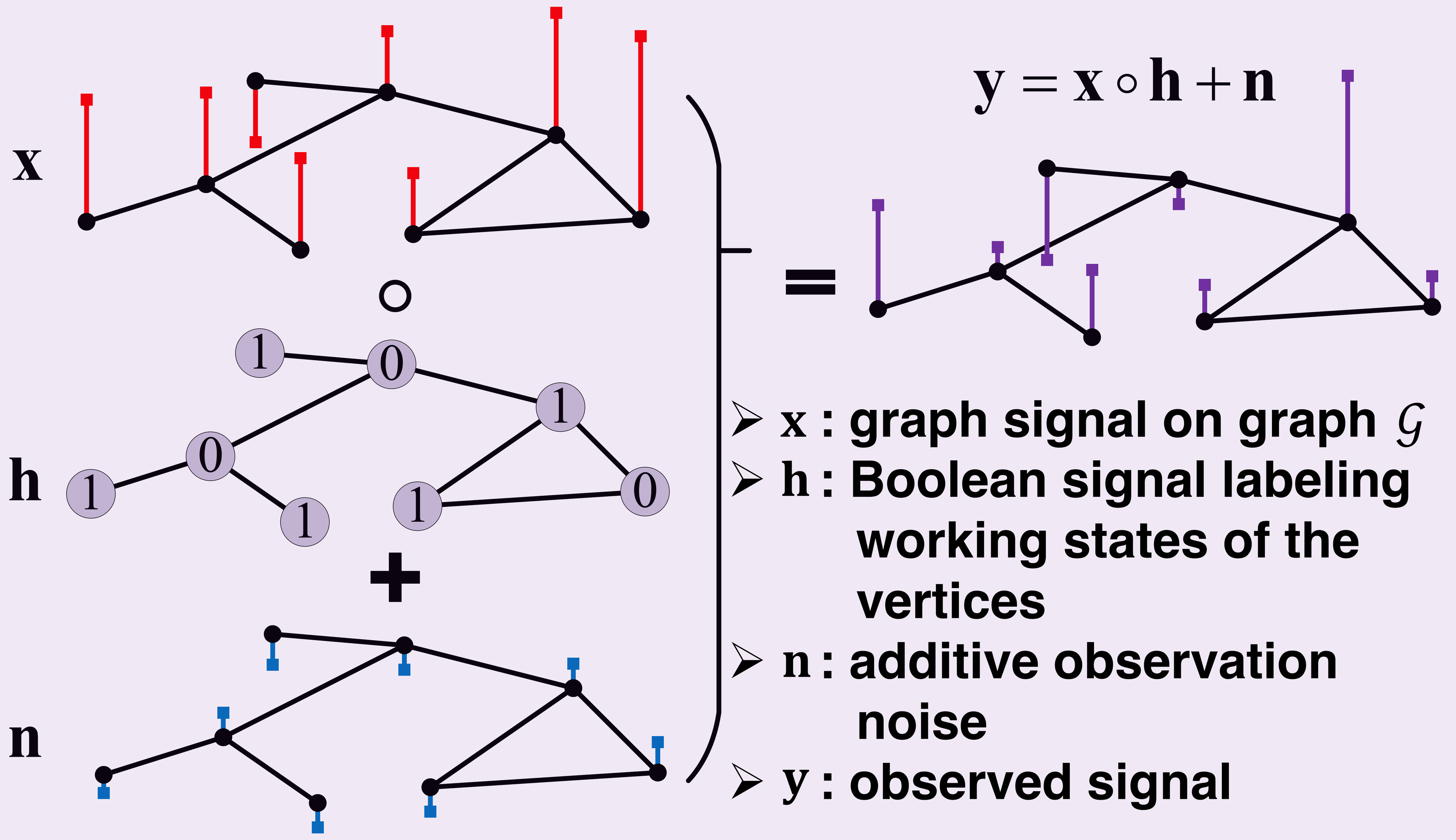
1 Problem & Motivation

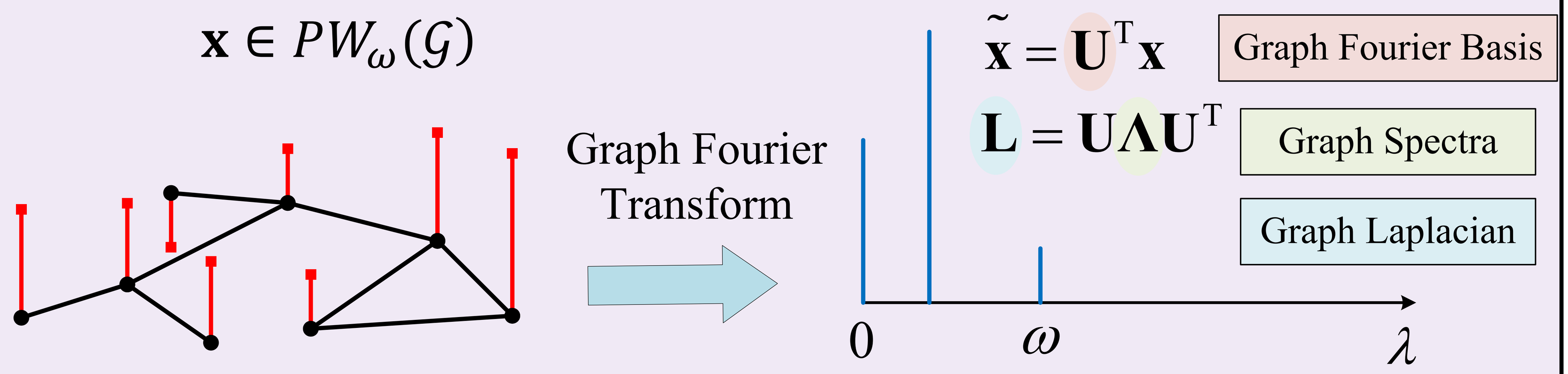
- **Motivation: Due to random defects of the measuring devices (vertices), the sampling states of the vertices are unknown.**
- **Problem: provided y, jointly detect h and reconstruct x.**



- **$\mathbf{x}$: graph signal on graph $\mathcal{G}$**
- **$\mathbf{h}$: Boolean signal labeling working states of the vertices**
- **$\mathbf{n}$: additive observation noise**
- **$\mathbf{y}$: observed signal**

2 Problem Modelling

- **Prior Information:**
  - **defects independent, vertex i breakdowns with probability p_i**
  - **Gaussian additive noise:**
  - **graph signal $\mathbf{x}$ is ω – bandlimited** $\mathbf{n} \sim \mathcal{N}(\mathbf{0}_N, \sigma^2 \mathbf{I}_N)$



3 Joint Detection and Reconstruction Algorithm

- **Maximum a posteriori (MAP) criterion:**

$$\max_{\mathbf{h},\mathbf{x}} P(\mathbf{h}|\mathbf{y};\mathbf{x}) \propto P(\mathbf{y}|\mathbf{h};\mathbf{x})P(\mathbf{h})$$

$$\text{s.t. } \mathbf{h} \in \{0,1\}^N, \mathbf{x} \in PW_\omega(\mathcal{G}).$$

- **Vertex-wise separability:**

$$P(\mathbf{y}|\mathbf{h};\mathbf{x})P(\mathbf{h}) \propto \prod_{i=1}^{N} \exp\left(-w_i h_i - \frac{(y_i - h_i x_i)^2}{2\sigma^2}\right), \quad w_i = \ln \frac{p_i}{1-p_i}$$

- **Optimization Problem:** nonconvex

$$\min_{\mathbf{h},\mathbf{x}} \ F(\mathbf{x},\mathbf{h}) := \mathbf{w}^T\mathbf{h} + \frac{\|\mathbf{y} - \mathbf{h}^\circ\mathbf{x}\|_2^2}{2\sigma^2} + \lambda\|\mathbf{x}\|_2^2 \quad (1)$$

$$\text{s.t. } \ \mathbf{x} \in PW_\omega(\mathcal{G}), \mathbf{h} \in \{0,1\}^N$$ integer

- **2 Subproblems:**

$$D(\mathbf{x},\mathbf{y}) = \underset{\mathbf{h}}{\operatorname{argmin}} F(\mathbf{x},\mathbf{h}), \quad \text{s.t. } \mathbf{h} \in \{0,1\}^N$$

$$R(\mathbf{h},\mathbf{y}) = \underset{\mathbf{x}}{\operatorname{argmin}} F(\mathbf{x},\mathbf{h}), \quad \text{s.t. } \mathbf{x} \in PW_\omega(\mathcal{G})$$

- **Alternating updating:**

1. **Detect defects:**

$$h_i^{k+1} = D_i(x_i^k, y_i) = \begin{cases} 1, & (x_i^k)^2 - 2x_i^k y_i < -2\sigma^2 w_i; \\ 0, & elsewhere. \end{cases}$$

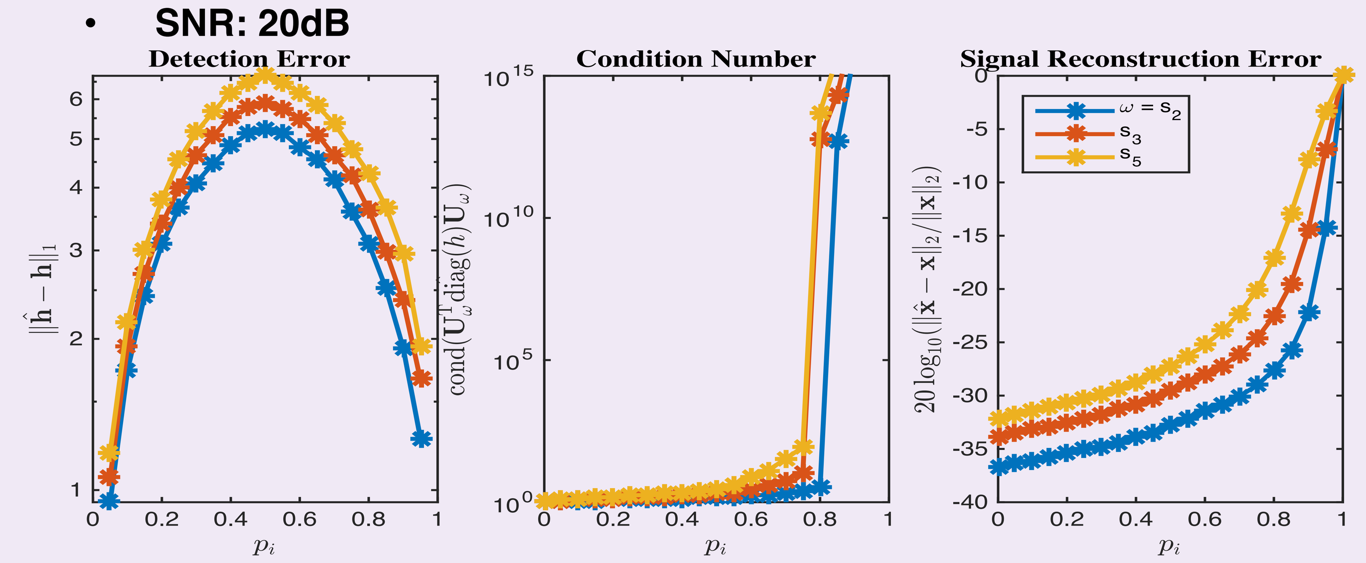
2. **Reconstruct graph signal:** $\mathbf{x}^{k+1} = R(\mathbf{h}^{k+1}, \mathbf{y})$
3. **Repeat 1, 2 until convergence**
4. **Output:** $(\hat{\mathbf{h}}, \hat{\mathbf{x}}) = (\mathbf{h}^k, \mathbf{x}^k)$

4 Theoretical Analysis: Balancing Guarantee

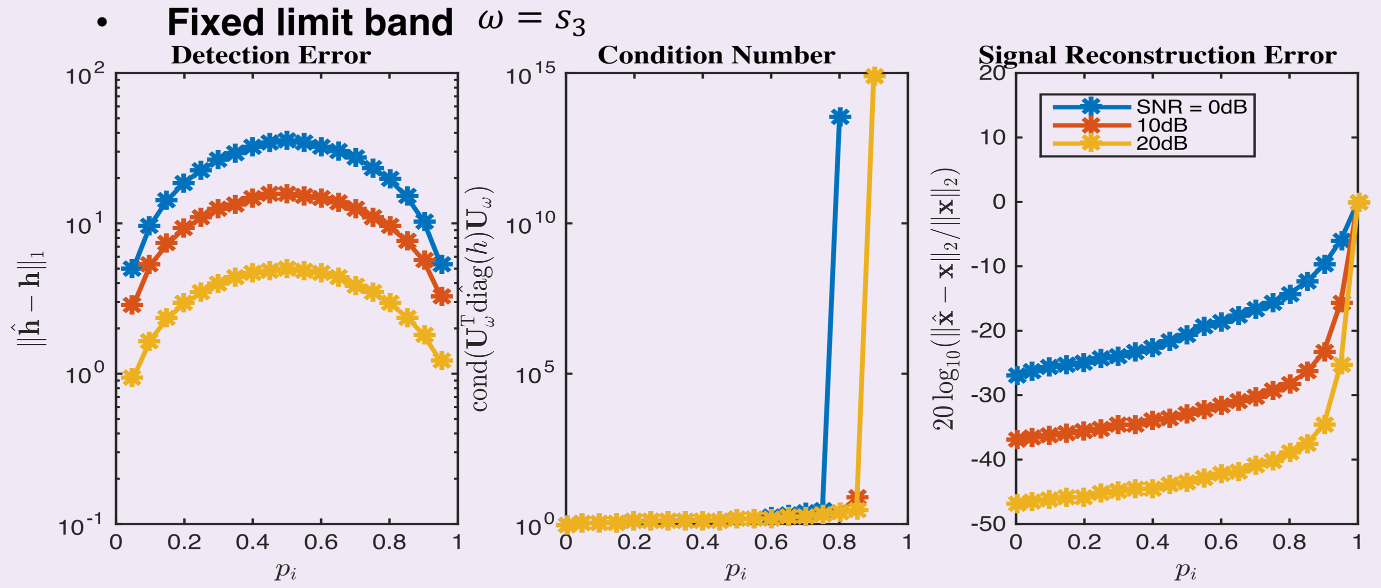
- **Convergence Property:**
  - **The sequence of objective function value $\{F(\mathbf{h}^k, \mathbf{x}^k)\}_{k\in\mathbb{N}}$ is non-increasing, and converges monotonically to a F^*.**
  - **The sequence of variables $\{\mathbf{h}^k, \mathbf{x}^k\}_{k\in\mathbb{N}}$ satisfies the following properties:**
    - **there is at least one accumulation point $(\mathbf{h}^*, R(\mathbf{h}^*, \mathbf{y}))$ of the sequence;**
    - **each accumulation point is a feasible solution to problem (1) with the same objective function value F^*.**
- **Confidence of Reconstruction Result:**
  - **matrix $\mathbf{U}_\omega^T diag(\mathbf{h})\mathbf{U}_\omega$ is invertible if and only if the working vertices compose a uniqueness set:**
    - **noiseless case: any ω- bandlimited graph signal can be uniquely reconstructed if the measurements on a uniqueness set is available.**
  - **the condition number of $\mathbf{U}_\omega^T diag(\mathbf{h})\mathbf{U}_\omega + 2\sigma^2\lambda\mathbf{I}_N$ measures the confidence of $\hat{\mathbf{x}}$.**

5 Simulation Result

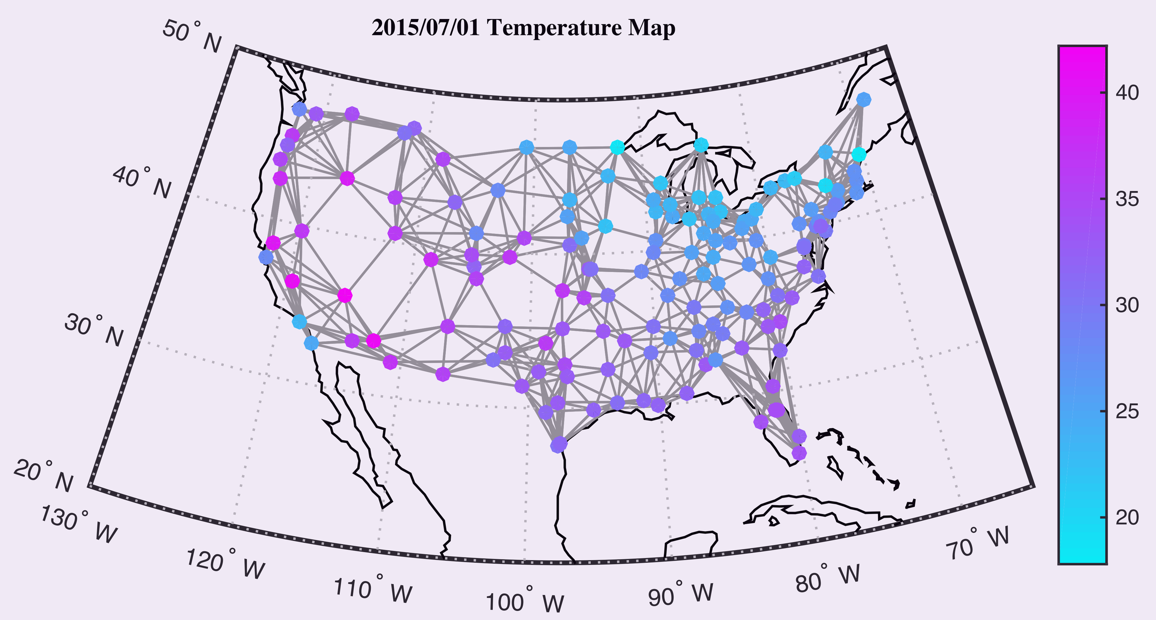
- **$\mathcal{G}$: 5- nearest neighbor graph of $N = 100$ randomly placed vertices**
- **$\mathbf{x}$: Gaussian random vector in $PW_\omega(\mathcal{G})$**
  - **SNR: 20dB**



- **Fixed limit band** $\omega = s_3$



- **$\mathcal{G}$: 8- nearest neighbor graph of $N = 150$ weather stations**
- **$\mathbf{x}$: projection of temperature signal onto $PW_{s_{10}}(\mathcal{G})$**



- **Comparison with Maximum-Likelihood (ML) method**
  - **SNR: 20dB**