Outage Probability Conjecture Does Not Hold for Two-Input-Multiple-Output (TIMO) System

Gen Li, Jingkai Yan, and Yuantao Gu
Beijing National Research Center for Information Science and Technology (BNRist)
Department of Electronic Engineering, Tsinghua University, Beijing 100084, China
Email: {g-li16, yjk14}@mails.tsinghua.edu.cn, gyt@tsinghua.edu.cn

***Abstract*—Multiple-Input-Multiple-Output (MIMO) communication systems have seen wide application due to its performance benefits such as multiplexing gain. For MIMO systems with non-ergodic Gaussian channel, a conjecture regarding its outage probability has been proposed by Telatar in [1] and long considered true. A special single-output case of this conjecture has been proved theoretically. In this work, we address the special Two-Input-Multiple-Output (TIMO) case, and show that the conjecture is untrue. A concrete counter-example is proposed and verified both theoretically and by numerical experiments. This result rejects the decades-long conjecture and provides interesting insight into the symmetry of MIMO systems.**

I. Introduction

Multi-Input-Multi-Output (MIMO) have been one of the core concepts and practices in modern communication systems since its pioneering proposal in the 1990s [2], [3]. With multiple transmitters and receivers, data rate and multi-user capabilities can be improved significantly [4], [5]. Current and future generations of communication systems all embrace MIMO as an essential part, including WLAN, LTE, 5G, and mmWave MIMO systems [6]–[10]. [11] reviews on the importance of MIMO in contemporary communication.

The theoretical channel capacity of MIMO systems is believed to be first addressed by Telatar in his work [1]. Capacity for mean (ergodic) Gaussian channel was solved by that work, and some later works addressed problems including parameter setting [12] and multi-user interference [13]. However, most existing works only address the channel capacity problem for mean (ergodic) Gaussian channels, and few works exist on the no less important non-ergodic channel [1]. For the capacity of non-ergodic MIMO channels, the author in [1] adopted a different approach as the outage probability, and proposed a conjecture on conditions for the minimum outage probability. This conjecture has long been assumed true and employed for the benchmark outage probability in later works [14], [15]. Proof of a special Multiple-Input-Single-Output (MISO) case of the conjecture was established in [15].

This paper approaches the Outage Probability Conjecture from a novel perspective, and provides a theoretical analysis of the particular Two-Input-Multiple-Output (TIMO) case. Our analysis reveals that the conjecture is actually false, with a concrete counter-example provided and verified by both theoretical derivation and numerical simulation [1], providing interesting insight into the symmetry of Gaussian MIMO channels.

The rest of this paper is organized as follows. Section II reviews the Outage Probability Conjecture and the proven case of MISO. Theoretical analysis of the TIMO case and the counter-example is given in Section III, which is the main contribution of our work. Numerical experiments are conducted in Section IV. Section V contains concluding remarks.

Notations: $\mathbf{H}^{\mathrm{T}}$ and $\mathbf{H}^*$ denote respectively the transpose and conjugate transpose of $\mathbf{H}$. $\#\mathcal{S}$ denotes the number of elements in set $\mathcal{S}$.

II. The Outage Probability Conjecture

A. The Conjecture

Consider a single-user MIMO Gaussian channel. Assume the number of transmitting antennas and receiving antennas is t and r, respectively. The channel model can be described as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{n}, \tag{1}$$

where $\mathbf{x} \in \mathbb{C}^t$, $\mathbf{y} \in \mathbb{C}^r$, and $\mathbf{n} \in \mathbb{C}^r$ denotes the transmitted vector, the received vector, and the additive noise, respectively. The channel is denoted by $\mathbf{H} \in \mathbb{C}^{r\times t}$, where each entry is *i.i.d.* circularly symmetric complex Gaussian variable with $\mathbb{E}\left|H_{i,j}\right|^2 = 1$.

In the non-ergodic case, the matrix $\mathbf{H}$ is random but is held fix once it is chosen. In this case, we consider the outage probability for evaluation of the channel. Let R be the data rate and P be the signal power constraint. The outage probability $P_{\mathrm{out}}(R, P)$ is defined as follows:

$$P_{\mathrm{out}}(R,P) := \inf_{\substack{\mathbf{Q}:\mathbf{Q}\succeq 0,\\ \mathrm{tr}(\mathbf{Q})\leq P}} \mathbb{P}[\log\det(\mathbf{I}_r + \mathbf{H}\mathbf{Q}\mathbf{H}^*) < R]. \tag{2}$$

In Telatar's words, $P_{\mathrm{out}}(R, P)$ is a probability such that *"For any rate less than R and any δ there exists a code satisfying the power constraint P for which the error probability is less than δ for all but a set of $\mathbf{H}$ whose total probability is less than $P_{\mathrm{out}}(R, P)$."* The conjecture is stated as the following [1]:

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[1] The supplementary downloadable material, including MATLAB codes for all experiments, is available at http://gu.ee.tsinghua.edu.cn/publications/.

Conjecture 1 (Outage Probability Conjecture) *The matrices* $\mathbf{Q}$ *that yield the infimum in* (4) *has eigenvalue decomposition* $\mathbf{UDU}^*$*, where* $\mathbf{U}$ *is* $t \times t$ *unitary and* $\mathbf{D}$ *has the form of*

$$\mathbf{D} = \frac{P}{k}\mathrm{diag}\big(\underbrace{1,\ldots,1}_{k},\underbrace{0,\ldots,0}_{t-k}\big), \tag{3}$$

where $1 \le k \le t$ *is some integer.*

The intuition behind this conjecture is the symmetry between the transmitters, where they are either working at some same power P/k or not working. This seems very plausible and therefore has been generally accepted as true.

While the above gives the original form of the Conjecture, in actuality we need only to consider the case of $\mathrm{tr}(\mathbf{Q}) = P$, as the outage probability decreases as maximum allowed power increases [15]. Therefore the model can be expressed as

$$P_{\mathrm{out}}(R,P) := \inf_{\substack{\mathbf{Q}:\mathbf{Q}\ge 0,\\ \mathrm{tr}(\mathbf{Q})=P}} \mathbb{P}[\log\det(\mathbf{I}_r + \mathbf{HQH}^*) < R]. \tag{4}$$

B. The Proven MISO Case

A special MISO case of the above conjecture was studied in [16] and [15], where the authors focused on the MISO case and offered a proof for the Conjecture with MISO. When $r = 1$, we are able to remove the determinant in (4), leading to the following conjecture:

Conjecture 2 (Outage Probability Conjecture for MISO) *Let*

$$\mathcal{D}(t) := \{\mathbf{Q} \in \mathbb{C}^{t\times t} \mid \mathbf{Q} \ge 0, \mathrm{tr}(\mathbf{Q}) \le 1\}$$

and $(H_i)_{1\le i\le t} \overset{i.i.d}{\sim} \mathcal{N}_{\mathbb{C}}(0,1)$*. For all* $x \in \mathbb{R}_+$*, there exists* $k \in \{1,\ldots,t\}$ *such that*

$$\arg\min_{\mathbf{Q}\in\mathcal{D}(t)} \mathbb{P}\{\mathbf{HQH}^* \le x\} = \left\{\mathbf{U}\mathrm{diag}\left(\underbrace{\frac{1}{k},\ldots,\frac{1}{k}}_{k},\underbrace{0,\ldots,0}_{t-k}\right)\mathbf{U}^* : \mathbf{U} \in \mathcal{U}(t)\right\}, \tag{5}$$

where $\mathcal{U}(t)$ *denotes the group of* $t \times t$ *unitary matrices.*

After *transforming the conjecture into the problem of Gaussian quadratic forms having largest tail probability corresponding to such diagonal matrices*, the proof of MISO case was completed. For specific details please refer to [15].

III. Main Result and Counter-Example

In this section, we present our analysis of the Two-Input-Multiple-Output case of the conjecture. A specific counter-example is demonstrated and verified.

The major difficulty of analyzing the general version of the Outage Probability Conjecture lies in the determinant on the right hand side (RHS) of (4). This also partially explains why the MISO case is comparatively earlier to solve than the general case. For the TIMO case, we adopt a new analytical approach to the simplification of the determinant.

With the special case of $t = 2$, we can write

$$\mathbf{Q} = \mathrm{diag}(q_1, q_2), \qquad \mathbf{H} = [\mathbf{h}_1, \mathbf{h}_2].$$

According to Sylvester's determinant theorem [17], we have

$$\begin{aligned}&\det(\mathbf{I}_r + \mathbf{HQH}^*) = \det(\mathbf{I}_r + \mathbf{H}^*\mathbf{HQ})\\ &=1+q_1|\mathbf{h}_1|^2+q_2|\mathbf{h}_2|^2+q_1q_2|\mathbf{h}_1|^2|\mathbf{h}_2|^2\left(1-\frac{|\mathbf{h}_1^*\mathbf{h}_2|^2}{|\mathbf{h}_1|^2|\mathbf{h}_2|^2}\right),\end{aligned} \tag{6}$$

By introducing three random variables S, T, and ρ, we observe the important fact that

$$\begin{aligned}S &:= q_1|\mathbf{h}_1|^2 \sim \frac{q_1}{2}\chi^2_{2r},\\ T &:= q_2|\mathbf{h}_2|^2 \sim \frac{q_2}{2}\chi^2_{2r},\\ 1-\rho &:= \frac{|\mathbf{h}_1^*\mathbf{h}_2|^2}{|\mathbf{h}_1|^2|\mathbf{h}_2|^2} \sim \mathrm{Beta}(1, r-1),\end{aligned}$$

where the probability density $f_1(x;k)$ for Chi-squared distribution χ^2_k and $f_2(x;\alpha,\beta)$ for Beta distribution $\mathrm{Beta}(\alpha,\beta)$ are, respectively,

$$f_1(x;k) = \begin{cases}\dfrac{x^{\frac{k}{2}-1}\mathrm{e}^{-\frac{x}{2}}}{2^{\frac{k}{2}}\Gamma\frac{k}{2}}, & x > 0;\\ 0, & \text{otherwise.}\end{cases}$$

$$f_2(x;\alpha,\beta) = \begin{cases}\dfrac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}, & 0 < x < 1;\\ 0, & \text{otherwise.}\end{cases}$$

Notice that these three random variables are independent of each other, as $\frac{|\mathbf{h}_1^*\mathbf{h}_2|^2}{|\mathbf{h}_1|^2|\mathbf{h}_2|^2}$ can be interpreted as the angle between vectors $\mathbf{h}_1$ and $\mathbf{h}_2$, and is therefore independent of the vector lengths.

Based on (6) and above interpretation, we could calculate the RHS of (4) by a probability integral. When $q_1, q_2 \ne 0$, the integral is

$$\begin{aligned}&\mathbb{P}[\log\det(\mathbf{I}_r + \mathbf{HQH}^*) < R]\\ =&\mathbb{P}[1 + S + T + ST\rho < \mathrm{e}^R]\\ =&\int_0^1 (r-1)\rho^{r-2}\mathrm{d}\rho\\ &\cdot\int_0^{\mathrm{e}^R-1}\frac{s^{r-1}\mathrm{e}^{-\frac{s}{q_1}}}{(r-1)!q_1^r}\int_0^{\frac{\mathrm{e}^R-1-s}{1+\rho s}}\frac{t^{r-1}\mathrm{e}^{-\frac{t}{q_2}}}{(r-1)!q_2^r}\mathrm{d}t\mathrm{d}s\\ =:&f(q_1,q_2)\end{aligned} \tag{7}$$

When q_1 or q_2 equals zero, assuming that $q_1 = P, q_2 = 0$ without loss of generality, the integral is

$$\mathbb{P}[\log\det(\mathbf{I}_r + \mathbf{HQH}^*) < R] = \int_0^{\mathrm{e}^R-1}\frac{s^{r-1}\mathrm{e}^{-\frac{s}{P}}}{(r-1)!P^r}\mathrm{d}s, \tag{8}$$

which can also be derived indirectly from (7) by taking the limit $q_1 = P$ and $q_2 \to 0$. This indicates that there is no need to make explicit distinction between the two cases, and we shall use the nonzero case in the following discussion when no ambiguity is possible.

According to the above analysis, the original conjecture is equivalent to the following:

Conjecture 3 (Outage Probability Conjecture for TIMO) *Let*

$$\mathcal{D} = \{(q_1, q_2) \in [0, P]^2 \mid q_1 + q_2 = P\}.$$

The pairs of (q_1, q_2) *that minimizes* $f(q_1, q_2)$ *in domain* $\mathcal{D}$ *fall in the set*

$$\left\{(P, 0), \left(\frac{P}{2}, \frac{P}{2}\right), (0, P)\right\}.$$

The RHS expression in (7), i.e. $f = f(q_1, q_2)$, is symmetric in the sense of $f(q_1, q_2) = f(q_2, q_1)$. It should be noted that while f has the form of a bivariate function, it is intrinsically univariate in our problem (4) due to the relation $q_1 + q_2 = P$.

As a specific instance, take the set of parameters

$$t = 2, \quad r = 2, \quad P = 0.5, \quad R = \ln 3.$$

Calculations of the first and second order derivatives at these special points give the following results:

$$\left.\frac{\mathrm{d}f}{\mathrm{d}q_1}\right|_{q_1=0} = 0, \quad \left.\frac{\mathrm{d}^2 f}{\mathrm{d}q_1^2}\right|_{q_1=0} = -\frac{8}{\mathrm{e}^4} < 0, \quad \left.\frac{\mathrm{d}f}{\mathrm{d}q_1}\right|_{q_1=\frac{P}{2}} = 0,$$
$$\left.\frac{\mathrm{d}^2 f}{\mathrm{d}q_1^2}\right|_{q_1=\frac{P}{2}} = \int_0^1 \int_0^2 \gamma(t, \rho)\mathrm{d}s\mathrm{d}\rho \approx -0.1014 < 0,$$

where

$$\gamma(s, \rho) = 4^6 s t^2 \mathrm{e}^{-4(s+t)} \left(2 - 8t + 8s\right), \quad t = \frac{2 - s}{1 + \rho s}.$$

Please see Appendix for detailed calculation. From the signs of the derivatives, there exists some q_m with $0 < q_m < \frac{P}{2}$ such that $(q_1, q_2) = (q_m, P - q_m)$ yields a smaller outage probability than as predicted by the Outage Probability Conjecture. A more detailed discussion can be found in [18].

Remark 1 *The above counter-example suggests that the seemingly intuitive symmetry in the Outage Probability Conjecture is actually misleading. One possible explanation is that the expression* (4) *in calculating outage probability is non-convex, and therefore does not enjoy many of the promising properties of convex functions.*

IV. Numerical Verification

In this section, we present both verification of the counter-example and experiments to test the Outage Probability Conjecture under various sets of parameter values.

A. Verification of the Counter-Example

Numerical verifications of the above counter-example is demonstrated in Fig. 1. We simulate both matrix-form Monte Carlo and probability integral, which yield identical results.

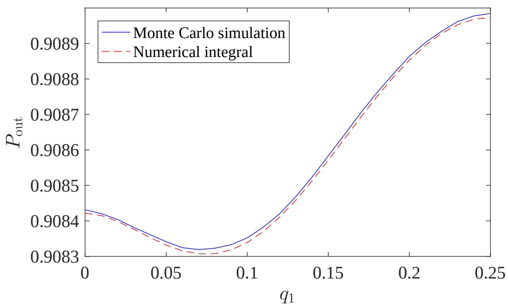

Fig. 1. Outage Probability plotted as a function of q_1 with parameters $t = 2, r = 2, P = 0.5, R = \ln 3$. The minimum is attained within the interval $(0.05, 0.1)$, instead of at 0 or $P/2$ as the Conjecture predicted.

B. Choice of Parameters

Of the four meta-parameters (t, r, R, P) in the MIMO system, we set $t = 2$ as a constant, and take different values for (r, R, P). With $t = 2$, testing the conjecture is equivalent to checking whether the minimizer of the outage probability (7) and (8) appears at the predicted points.

For simplicity we set $r = 2$ fixed, and let R and P vary. We first conduct a broad search, where the channel rate R ranges from $0.05r$ to $5r$ at step $0.05r$, and the power P range from $0.05r$ to $5r$ at step $0.05r$. We need to look for the $q_m \in \left[0, \frac{P}{2}\right]$ that achieves the smallest outage probability among these sample values. When neither 0 nor $\frac{P}{2}$ minimizes the expression, it is safe to conclude that the conjecture is false. Results are shown in the top plot in Fig. 2, which only identifies one region where the conjecture fails. A thick red dot indicates a (R, P) set where we have sufficient evidence that the conjecture failed, and a blue dot indicates where the conjecture appears correct at least from our sample points. Data in the bottom-right blank area is discarded, because the outage probability becomes too close to 1 for stable numerical calculation. It is also safe to say those cases are meaningless is reality, because the channel would be useless.

Zooming in the parameters for a finer examination, we then let R range from $0.42r$ to $0.7r$ at step $0.02r$, and P from $0.2r$ to $0.3r$ at step $0.02r$. q is set at sample values from 0 to $0.5P$ at step $0.025P$. This result is plotted in the bottom plot in Fig. 2.

The above experiments reveal that the Outage Probability Conjecture is likely true under most circumstances, which partly explains why it has been trusted for so long. In fact, we have also tested the conjecture for $r = 3$ and $r = 4$ with a broad range of values for (R, P), and surprisingly did not identify any counter-example. This opens up possibilities for future work on the general condition for the Conjecture.

V. Conclusion

In this paper, we present the theoretical analysis of the Outage Probability Conjecture regarding non-ergodic MIMO channel. By focusing on the two-transmitter case, we are able to reformulate the original Conjecture into a probability

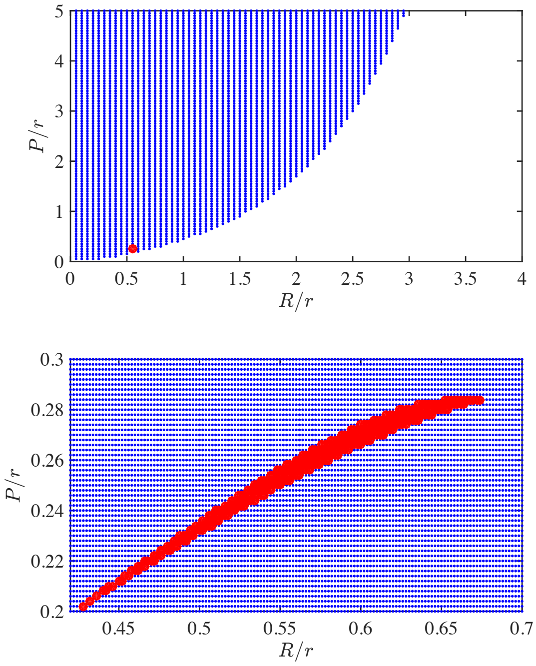

Fig. 2. Test of the Conjecture with $t = r = 2$ and (R, P) varying. The Conjecture is mostly true with few exceptions. The bottom plot specifies the exception region in the top plot, where small blue dot and big red dot denotes the true and false sample, respectively.

integral, allowing for direct analysis. A concrete counter-example is shown to reject the long-believed Conjecture. We verify the counter-example by both theoretical derivation and numerical experiment. Experiments are conducted to test the Conjecture on different parameter sets, revealing that the Conjecture is true in most cases, despite the counter-example identified here. Future directions extending this work could focus on specific conditions for the Conjecture to fail, or the analysis of the more general case with arbitrary number of transmitters.

APPENDIX

Here we calculate the first and the second order derivatives of the expression in (7).

We start by reiterating the fact that $f = f(q_1, q_2)$ is actually a univariate function with constraint $q_1 + q_2 = P$. We will use the derivative such as $\mathrm{d}f/\mathrm{d}q_1$ to denote the derivative of f as a univariate function with the constraint, and use the partial derivative such as $\partial f(q_1, q_2)/\partial q_1$ to denote the partial derivative of f as a bivariate function without any constraint. Such notation would facilitate our discussion.

Due to symmetry between q_1 and q_2 in the expression, the partial derivative can be calculated in the following terms:.

$$\frac{\mathrm{d}f}{\mathrm{d}q_1} = \frac{\partial f(q_1,q_2)}{\partial q_1} - \frac{\partial f(q_1,q_2)}{\partial q_2} = \frac{\partial f(q_2,q_1)}{\partial q_1} - \frac{\partial f(q_1,q_2)}{\partial q_2}, \tag{9}$$

$$\frac{\mathrm{d}^2 f}{\mathrm{d}q_1^2} = \frac{\partial^2 f(q_1,q_2)}{\partial q_1^2} + \frac{\partial^2 f(q_1,q_2)}{\partial q_2^2} - 2\frac{\partial^2 f(q_1,q_2)}{\partial q_1 \partial q_2}$$
$$= \frac{\partial^2 f(q_2,q_1)}{\partial q_1^2} + \frac{\partial^2 f(q_1,q_2)}{\partial q_2^2} - 2\frac{\partial^2 f(q_1,q_2)}{\partial q_1 \partial q_2}. \tag{10}$$

Let us begin from calculating the first order derivatives. Denote $u = \mathrm{e}^R - 1$ for briefness. When $q_1, q_2 \neq 0$,

$$\frac{\partial f(q_1,q_2)}{\partial q_2} = \int_0^1 (r-1)\rho^{r-2}\mathrm{d}\rho \cdot A, \tag{11}$$

where

$$\begin{aligned} A &:= \int_0^u \frac{s^{r-1}\mathrm{e}^{-\frac{s}{q_1}}}{(r-1)!q_1^r} \cdot \frac{\partial \int_0^{\frac{1}{q_2}\frac{u-s}{1+\rho s}} \frac{t^{r-1}\mathrm{e}^{-t}}{(r-1)!}\mathrm{d}t}{\partial q_2}\mathrm{d}s \\ &= -\int_0^u \frac{s^{r-1}\left(\frac{u-s}{1+\rho s}\right)^r}{(r-1)!^2 q_1^r q_2^{r+1}} \mathrm{e}^{-\frac{s}{q_1}-\frac{1}{q_2}\frac{u-s}{1+\rho s}}\mathrm{d}s \\ &= -\int_0^u \frac{s^r\left(\frac{u-s}{1+\rho s}\right)^{r-1}}{(r-1)!^2 q_1^r q_2^{r+1}} \frac{1+\rho u}{(1+\rho t)^2} \mathrm{e}^{-\frac{s}{q_2}-\frac{1}{q_1}\frac{u-s}{1+\rho s}}\mathrm{d}s. \end{aligned}$$

Since $f(q_1,q_2) = f(q_2,q_1)$, we have $\frac{\partial f(q_1,q_2)}{\partial q_1} = \frac{\partial f(q_2,q_1)}{\partial q_1}$. Therefore (11) can be used to calculate the first order derivatives at any point except for 0 and P.

In addition, when one of the variables is zero, we have

$$\left.\frac{\partial f(q_1,q_2)}{\partial q_2}\right|_{\substack{q_1=0,\\ q_2=P}} = -\frac{u^r\mathrm{e}^{-\frac{u}{P}}}{(r-1)!P^{r+1}}$$

and

$$\left.\frac{\partial f(q_1,q_2)}{\partial q_2}\right|_{\substack{q_1=P,\\ q_2=0}} = -\frac{u^{r-1}\mathrm{e}^{-\frac{u}{P}}}{(r-1)!P^r}(r+(r-1)u).$$

Next we calculate the second order derivatives. When $q_1, q_2 \neq 0$, we have

$$\frac{\partial^2 f(q_1,q_2)}{\partial q_2^2} = -\int_0^1 (r-1)\rho^{r-2}\mathrm{d}\rho \cdot B, \tag{12}$$

where

$$\begin{aligned} B &:= \int_0^u \frac{\partial}{\partial q_2} \frac{s^{r-1}\left(\frac{u-s}{1+\rho s}\right)^r}{(r-1)!^2 q_1^r q_2^{r+1}} \mathrm{e}^{-\frac{s}{q_1}-\frac{1}{q_2}\frac{u-s}{1+\rho s}}\mathrm{d}s \\ &= -\int_0^u \frac{s^{r-1}\left(\frac{u-s}{1+\rho s}\right)^r}{(r-1)!^2 q_1^r q_2^{r+2}} \mathrm{e}^{-\frac{s}{q_1}-\frac{1}{q_2}\frac{u-s}{1+\rho s}} \\ &\qquad \cdot \left(r+1-\frac{1}{q_2}\frac{u-s}{1+\rho s}\right)\mathrm{d}s, \end{aligned}$$

and

$$\frac{\partial^2 f(q_1,q_2)}{\partial q_1 \partial q_2} = -\int_0^1 (r-1)\rho^{r-2}\mathrm{d}\rho \cdot C, \tag{13}$$

where

$$\begin{aligned} C &:= \int_0^u \frac{\partial}{\partial q_1} \frac{s^{r-1}\left(\frac{u-s}{1+\rho s}\right)^r}{(r-1)!^2 q_1^r q_2^{r+1}} \mathrm{e}^{-\frac{s}{q_1}-\frac{1}{q_2}\frac{u-s}{1+\rho s}}\mathrm{d}s \\ &= -\int_0^u \frac{s^{r-1}\left(\frac{u-s}{1+\rho s}\right)^r}{(r-1)!^2 q_1^{r+1} q_2^{r+1}} \mathrm{e}^{-\frac{s}{q_1}-\frac{1}{q_2}\frac{u-s}{1+\rho s}}\left(r-\frac{s}{q_1}\right)\mathrm{d}s. \end{aligned}$$

Equations (12) and (13) can be used to calculate the second order derivatives at any point except for 0 and P.

In addition, when one of the variables is zero, we have

$$\left.\frac{\partial^2 f(q_1,q_2)}{\partial q_2^2}\right|_{\substack{q_1=0,\\ q_2=P}} = \frac{u^r \mathrm{e}^{-\frac{u}{P}}}{(r-1)!P^{r+2}}\left(r+1-\frac{u}{P}\right),$$

$$\left.\frac{\partial^2 f(q_1,q_2)}{\partial q_2^2}\right|_{\substack{q_1=P\\ q_2=0}} = \frac{u^r \mathrm{e}^{-\frac{u}{P}}}{(r-1)!P^{r+1}} r(r+1) \cdot\left(P(r-1)(\frac{1}{u}+1)^2-\frac{1}{u}-\frac{2(r-1)}{r}-\frac{u(r-1)}{r+1}\right),$$

$$\left.\frac{\partial^2 f(q_1,q_2)}{\partial q_1\partial q_2}\right|_{\substack{q_1=0,\\ q_2=P}} = \frac{u^r \mathrm{e}^{-\frac{u}{P}}}{(r-1)!P^{r+2}}\left(\frac{rP}{u}-1\right)(r+(r-1)u),$$

and

$$\left.\frac{\partial^2 f(q_1,q_2)}{\partial q_1\partial q_2}\right|_{\substack{q_1=P\\ q_2=0}} = \left.\frac{\partial^2 f(q_1,q_2)}{\partial q_1\partial q_2}\right|_{\substack{q_1=0,\\ q_2=P}}.$$

By plugging these partial derivatives into (9) and (10), and take the values $t=2, \quad r=2, \quad P=0.5, \quad R=\ln 3$, we readily obtain the first and second order derivatives at $(0,P)$ and $(\frac{P}{2},\frac{P}{2})$.

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