A Non-convex Approach for Sparse Recovery with
Convergence Guarantee

Laming Chen and Yuantao Gu∗

Received December 24, 2012, revised May 22, 2013.

Abstract

In the area of sparse recovery, numerous researches hint that non-convex penalties might induce better sparsity than convex ones, but up until now the non-convex algorithms lack convergence guarantee from the initial solution to the global optimum. This paper aims to provide theoretical guarantee for sparse recovery via non-convex optimization. The concept of weak convexity is incorporated into a class of sparsity-inducing penalties to characterize their non-convexity. It is shown that in a neighborhood of the sparse signal (with radius in inverse proportion to the non-convexity), any local optimum can be regarded as a stable solution. It is further proved that if the non-convexity of the penalty function is below a threshold, the initial solution also belongs to this neighborhood. In addition, The idea of projected (sub)gradient method is generalized to solve this non-convex optimization problem. A uniform approximate projection can also be applied in the projection step to make the algorithm computationally tractable for large scale problems. The theoretical convergence analysis of these methods is provided in the noisy scenario. The result reveals that if the non-convexity is below a threshold, these methods would converge from the initial solution, and the recovered solution is with recovery error linear in both the noise term and the step size. Numerical simulations are performed to test the performance of the proposed approach and verify the theoretical analysis.

Keywords: Sparse recovery, sparseness measure, weakly convex, non-convex optimization, projected generalized gradient (PGG), approximate PGG (APGG), approximate pseudo-inverse matrix, convergence analysis.

1 Introduction

Since the introduction of compressive sensing (CS) [1–3], sparse recovery has received much attention and becomes a very hot topic these years [4–8]. Sparse recovery aims to solve the

∗This work was partially supported by National Natural Science Foundation of China (NSFC 60872087 and NSFC U0835003). The authors are with Department of Electronic Engineering, Tsinghua University, Beijing 100084, China. The corresponding author of this paper is Yuantao Gu (Email: gyt@tsinghua.edu.cn).
following underdetermined linear system

\[ \mathbf{y} = \mathbf{Ax}, \quad (1) \]

where \( \mathbf{y} \in \mathbb{R}^M \) denotes the measurement vector, \( \mathbf{A} \in \mathbb{R}^{M \times N} \) is a sensing matrix with more columns than rows, i.e., \( M < N \), and \( \mathbf{x} = (x_i) \in \mathbb{R}^N \) is the sparse or compressible signal to be recovered.

Many novel algorithms have been proposed to solve the problem (1). If \( \mathbf{x} \) is sparse, one typical method is to solve the following optimization problem

\[
\underset{\mathbf{x}}{\text{argmin}} \|\mathbf{x}\|_0 \quad \text{subject to} \quad \mathbf{y} = \mathbf{Ax},
\]

(2)

where the \( \ell_0 \) “norm” \( \|\mathbf{x}\|_0 = \#\{i : x_i \neq 0\} \) counts the non-zero elements of \( \mathbf{x} \). However, it is not practical to adopt this method since \( \ell_0 \)-minimization (2) is usually solved by combinatorial search, which is NP-hard. An alternate method [9] is to replace the \( \ell_0 \) “norm” with the \( \ell_1 \) norm, i.e.,

\[
\underset{\mathbf{x}}{\text{argmin}} \|\mathbf{x}\|_1 \quad \text{subject to} \quad \mathbf{y} = \mathbf{Ax}.
\]

(3)

The convex optimization problem (3) is also known as basis pursuit (BP). It is certified that under some certain conditions [10], the optimal solution of \( \ell_1 \)-minimization (3) is identical to that of (2). This conclusion greatly reduces the computational complexity, since (3) can be reformulated as a linear program (LP), and be solved by numerous efficient algorithms [11].

Another family of sparse recovery algorithms, greedy pursuit, is also proposed with the advantages of intuitive interpretation and low computational complexity. These algorithms iteratively draw the locations of non-zero elements of \( \mathbf{x} \), and then estimate the sparse signal by least squares (LS). Orthogonal matching pursuit (OMP) [12,13], which is a typical greedy algorithm, selects one more index in each iteration. Several improved algorithms based on OMP include regularized OMP (ROMP) [14], stagewise OMP (StOMP) [15], compressive sampling matching pursuit (CoSaMP) [16,17], and subspace pursuit (SP) [18].

Besides the above two well-known classes, another family of sparse recovery algorithms is put forward based on non-convex optimization

\[
\underset{\mathbf{x}}{\text{argmin}} J(\mathbf{x}) \quad \text{subject to} \quad \mathbf{y} = \mathbf{Ax},
\]

(4)

where \( J(\cdot) \) is a sparsity-inducing penalty. These algorithms include focal underdetermined system solver (FOCUSS) [19], iteratively reweighted least squares (IRLS) [20], reweighted \( \ell_1 \)-minimization [21], smoothed \( \ell_0 \) (SL0) [22], DC Algorithm [50], improved smoothed \( \ell_0 \) (ISL0) [23], and zero-point attracting projection (ZAP) [24]. It is theoretically proved [25–28] and experimentally verified [19–25,27,28] that for some certain non-convex penalties \( J(\cdot) \), (4) tends to derive the sparsest solution with looser conditions than (3). However, the inherent deficiency of multiple local minima in non-convex optimization limits its practical usage, where improper initial criteria might cause the solution trapped into them.
This paper aims to provide theoretical convergence guarantee for non-convex approaches from the initial solution to the global optimum. The question, which naturally appears and mainly motivates this paper, is raised as follows.

**Question.** Does there exist an algorithm, and in what circumstances, that guarantees to find the sparsest solution to the non-convex optimization (4)?

In this paper, exploiting the concept of *weak convexity* [29] to characterize the non-convexity of penalty functions, the mentioned question is replied as follows.

**Answer.** A non-convex approach is proposed with guarantee that it converges to the sparsest solution provided that the non-convexity of the penalty function is below a threshold.

In this paper, the idea of projected (sub)gradient method [30–33] is generalized to solve the non-convex optimization problem (4). For a class of sparsity-inducing penalties combining the concepts of sparseness measure [26] and weak convexity [29], the generalized gradients can be applied as the step direction. Depending on whether the projection step is accurate or not, the methods are termed *projected generalized gradient* (PGG) or *approximate PGG* (APGG), respectively. The theoretical convergence analysis of these methods in the noisy scenario is demonstrated in this paper. It reveals that as long as the non-convexity of the penalty function is below a *threshold* (which is in inverse proportion to the distance between the initial solution and the sparse signal), the iterative solution will get into the neighborhood of the desired sparse signal with radius linear in both the noise term and the step size. In the noiseless scenario, with sufficiently small step size, these methods will return a solution with any given precision.

The paper is organized as follows. Section 2 introduces the mathematical framework of this paper, including the weakly convex sparseness measures, the sparsity-inducing penalties, the PGG and APGG methods. Some related state of the art results are briefly reviewed in Section 3. In Section 4, the theoretical convergence analysis and discussions of the PGG and APGG methods are demonstrated. Numerical simulations are performed in Section 5 to verify the theoretical results. All of the proofs of the theoretical results are included in Section 6, and this paper is concluded in Section 7.

## 2 Mathematical Framework

### 2.1 Weakly Convex Sparseness Measures

First, a class of sparsity-inducing penalties is introduced with characterization of non-convexity. The penalty $J(x)$ in (4) is defined as

$$J(x) = \sum_{i=1}^{N} F(x_i),$$

(5)
where $F(\cdot)$ belongs to a class of sparseness measures satisfying the following Definition 1. Under this circumstances, we term the optimization problem (4) as $J$-minimization. The definitions and properties of weakly convex function $F(\cdot)$ with its generalized gradient set $\partial F(\cdot)$ can be found in [29,36].

**Definition 1** The function $F : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the following properties:

1. $F(0) = 0$, $F(\cdot)$ is even and not identically zero;
2. $F(\cdot)$ is non-decreasing on $[0, +\infty)$;
3. The function $t \mapsto F(t)/t$ is non-increasing on $(0, +\infty)$;
4. $F(\cdot)$ is a weakly convex function on $[0, +\infty)$.

Definition 1 is essentially a combination of the concepts of sparseness measures [26] and weak convexity. As has been revealed in [26], the concepts of null space property (NSP) and its constant [35] are closely related to whether the problem (4) is able to find the desired sparse signal. Define $x_S$ as the vector generated by setting the entries of $x$ indexed by $S^c = \{1, 2, \ldots, N\} \setminus S$ to zeros.

**Definition 2** Define null space constant $\gamma(J, A, K)$ as the smallest number such that

$$
J(z_S) \leq \gamma(J, A, K)J(z_{S^c})
$$

holds for all set $S$ with $\#S \leq K$ and for all vector $z \in \mathcal{N}(A)$, where $\mathcal{N}(A)$ denotes the null space of $A$.

The following proposition is derived in [26]. It reveals that the null space constant is a somewhat tight quantity for tuple $(J, A, K)$ indicating the performance of the problem (4), and that the performance of $J$-minimization lies between $\ell_1$-minimization and $\ell_0$-minimization.

**Proposition 1** (Theorem 2, 3, and 5 from [26]). For penalty $J(\cdot)$ formed by $F(\cdot)$ satisfying Definition 1.1)-3), the following statements hold:

1. If $\gamma(J, A, K) < 1$, then for all $x$ satisfies $\|x\|_0 \leq K$ and $y = Ax$, $x$ is the unique solution to (4);
2. If $\gamma(J, A, K) > 1$, then there exist $x$ and $x'$ such that $\|x\|_0 \leq K$, $Ax = Ax'$ and $J(x') < J(x)$;
3. $\gamma(\ell_0, A, K) \leq \gamma(J, A, K) \leq \gamma(\ell_1, A, K)$. 
Table 1: Weakly Convex Sparseness Measures with Parameter $\rho$
(with Parameter Requirements $0 \leq p < 1$ and $\sigma > 0$)

<table>
<thead>
<tr>
<th>No.</th>
<th>$F(t)$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>$</td>
<td>t</td>
</tr>
<tr>
<td>2.</td>
<td>$\frac{</td>
<td>t</td>
</tr>
<tr>
<td>3.</td>
<td>$1-e^{-\sigma</td>
<td>t</td>
</tr>
<tr>
<td>4.</td>
<td>$\ln(1+\sigma</td>
<td>t</td>
</tr>
<tr>
<td>5.</td>
<td>$\tan(\sigma</td>
<td>t</td>
</tr>
<tr>
<td>6.</td>
<td>$2\sigma</td>
<td>t</td>
</tr>
</tbody>
</table>

Figure 1: The sparseness measures in TABLE 1 are plotted. The parameter $p$ is set to 0.5. The parameters $\sigma$ are set respectively so that they all contain the point (0.9, 0.9).

Besides the definition of sparseness measures, an additional requirement Definition 1.4) is imposed on $F(\cdot)$ to characterize its non-convexity. In a nutshell, weakly convex function $F(\cdot)$ can be decomposed to $F(t) = H(t) + \rho t^2$, where $\rho < 0$ is the largest number such that $H(\cdot)$ is convex. Some commonly used weakly convex sparseness measures in [21, 24, 37–39] are listed in TABLE 1 and plotted in Fig. 1, where $X_P$ denotes the indicator function

$$X_P = \begin{cases} 1 & \text{P is true;} \\ 0 & \text{P is false.} \end{cases}$$

It needs to be emphasized that the penalty $J(\cdot)$ in this paper cannot be the widely used $\ell_p$ “norm” $(0 \leq p < 1)$ penalty in the literatures of sparse recovery [25, 27]. The function

$$F(t) = |t|^p \quad p \in [0, 1)$$

(7)
satisfies Definition 1.1)-3), but goes against the requirement of weak convexity, i.e., Definition 1.4). However, approximations to (7) are usually introduced to avoid infinite derivative around zero point and to improve the robustness. For example, (7) is approximated by

\[ F(t) = \frac{|t|}{(|t| + \sigma)^{1-p}} \]

with \( \sigma > 0 \) in [27]. The approximation now satisfies Definition 1, and its parameter \( \rho \) is shown in TABLE 1. This hints that weak convexity is reasonable and is an implicated assumption when some robust algorithms or theoretical analysis is taken into consideration.

2.2 (Approximate) Projected Generalized Gradient Method

Many algorithms can be adopted to solve the non-convex problem (4). In this paper, we focus on the well known gradient-based algorithm which is relatively easy to provide its convergence guarantee even when the problem is not convex.

Mathematically, the method is described as follows. Initialized as the least squares solution \( \mathbf{x}(0) = \mathbf{A}^\dagger \mathbf{y} \) where \( \mathbf{A}^\dagger = \mathbf{A}^T (\mathbf{A} \mathbf{A}^T)^{-1} \) denotes the pseudo-inverse matrix of \( \mathbf{A} \), the iterative solution \( \mathbf{x}(n) \) obeys

\[ \tilde{\mathbf{x}}(n+1) = \mathbf{x}(n) - \kappa \nabla J(\mathbf{x}(n)), \tag{8} \]
\[ \mathbf{x}(n+1) = \tilde{\mathbf{x}}(n+1) + \mathbf{A}^\dagger (\mathbf{y} - \mathbf{A} \tilde{\mathbf{x}}(n+1)) \tag{9} \]

where \( \kappa > 0 \) denotes the step size and \( \nabla J(\mathbf{x}) \) is a column vector whose \( i \)-th element is \( f(x_i) \in \partial F(x_i) \). Since the generalized gradient is adopted to update the iterative solutions, following the projected gradient method [30] and the projected subgradient method [31], this method is termed projected generalized gradient (PGG) method. The procedure of this method is described in TABLE 2. The iteration stops when the iterative number exceeds a certain bound.

The projection step of the PGG method involves the pseudo-inverse matrix \( \mathbf{A}^\dagger \), while exact calculation of it may be computationally intractable or even impossible because of its large scale in practical applications. To reduce the computational burden, a uniform approximate pseudo-inverse matrix of \( \mathbf{A} \) can be adopted. This method is termed approximate PGG (APGG) method. According to Appendix A which introduces approximate calculation of the pseudo-inverse matrix, let \( \mathbf{A}^T \mathbf{B} \) denote the approximation of \( \mathbf{A}^\dagger \). To characterize the approximate precision of the pseudo-inverse matrix, define

\[ \|\mathbf{I} - \mathbf{A} \mathbf{A}^T \mathbf{B}\|_2 \leq \zeta \]

where \( \| \cdot \|_2 \) denotes the spectral norm of the matrix, and we assume that \( \zeta < 1 \) throughout this paper.
Table 2: The Procedure of the PGG Method

<table>
<thead>
<tr>
<th>Input: A, y;</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initialization:</td>
</tr>
<tr>
<td>Calculate $A^\dagger$, $x(0) = A^\dagger y$, $n = 0$;</td>
</tr>
<tr>
<td>Output: x(n).</td>
</tr>
<tr>
<td>Repeat:</td>
</tr>
<tr>
<td>Generalized gradient step:</td>
</tr>
<tr>
<td>Update iterative solution by (8);</td>
</tr>
<tr>
<td>Projection step:</td>
</tr>
<tr>
<td>Update iterative solution by (9);</td>
</tr>
<tr>
<td>Iterative number increases by one:</td>
</tr>
<tr>
<td>$n = n + 1$;</td>
</tr>
<tr>
<td>Until: Stopping criterion satisfied;</td>
</tr>
</tbody>
</table>

3 State of the Art

Before formally introducing the main results of our paper, some related state of the art researches are introduced. Being aware of them might be of benefit in realizing the contributions of our paper.

Since the introduction of the concept of weak convexity [53], a branch of research work has been focused on the duality and optimality conditions for weakly convex minimization problem, for example [54–56]. The research can be regarded as the extension of those in convex optimization. They mainly consider the condition under which a point is the global minimizer of a weakly convex problem, which differs from the goal of our paper: providing convergence guarantee of a specific algorithm. In the area of sparse recovery, little attention has been paid to the concept of weak convexity. Our paper can be considered as an attempt to introduce the concept of weak convexity to the field of compressive sensing and sparse recovery, and there is still much room for further research.

The convergence performance of some non-convex sparse recovery algorithms has been studied in some literatures. For example, in [43], the solution sequence of IRLS [20] for $\ell_1$-minimization is proved to converge from the LS solution to the sparse one, while for $\ell_p$-minimization with $p \in (0, 1)$, a local convergence result is established where the convergence is guaranteed in a sufficiently small neighborhood of the sparse signal. Whether or not this neighborhood contains the initial solution is not discussed. In [57], the MM subspace algorithm is proposed to solve the $\ell_2$-$\ell_0$ regularized problem and its convergence performance is also provided. Under some assumptions, it is shown that the generated sequence will converge to a critical point, which is not, however, proved to be the global optimum. In [58],
the convergence performance of SL0 [22] is given. This is done due to the "local convexity" of the penalties, which are quite different from the penalties in our work. Besides, SL0 needs to solve a sequence of optimization problems to guarantee convergence to the sparse signal, while in our problem setting solving only one optimization problem is sufficient.

Recently some theoretical progress is made based on the projected subgradient method. In [41], the inexact projections are adopted, but unlike our work, these projections require approaching the exact one in the course of the algorithm. Another approximate subgradient projection method is introduced in [42]. Rather than approximate projection, it considers approximate subgradient. The ZAP algorithm [24] is essentially a special case of PGG, which adopts the No. 6 sparseness measure in TABLE 1. The literature [34] attempts to provide the convergence analysis of ZAP, yet the analysis is only for \(\ell_1\)-ZAP which uses the convex \(\ell_1\) norm as the sparsity-inducing penalty. Despite of this fact, it already contains some important ideas in the theoretical analysis of this paper.

To verify the theoretical analysis in the main text, numerical simulations are taken in the setting of random Gaussian sensing matrices. We have noticed that there is previous research characterizing the precise behavior of general penalization terms with Gaussian sensing matrix. One may read [59] for further reference.

\section{Convergence Analysis}

In this section, the main results on the convergence of the PGG and APGG methods are demonstrated. In the following, consider the noisy scenario \(y = Ax^* + e\) where \(x^*\) is the \(K\)-sparse signal to be recovered with \(T = \{i : x^*_i \neq 0\}\) as its support set.

To begin with, several properties of \(F(\cdot)\) are revealed in the following lemma, which is quite helpful in the proofs of the main results. Define \(\partial F(0) = \{0\} \).

**Lemma 1** The weakly convex sparseness measure \(F(\cdot)\) satisfies the following properties:

1. For all \(t_1, t_2 \in \mathbb{R}\), \(F(t_1 + t_2) \leq F(t_1) + F(t_2)\);

2. For all \(t \in (0, +\infty)\) and \(f(t) \in \partial F(t)\), \(f(t) \geq 0\);

3. \(F(\cdot)\) is continuous and there exists \(\alpha > 0\) such that \(F(t) \leq \alpha |t|\) holds for all \(t \in \mathbb{R}\);

4. For all \(t \in \mathbb{R}\) and \(f(t) \in \partial F(t)\), \(|f(t)| \leq \alpha\);

5. For all \(t_1, t_2 \in \mathbb{R}\) and \(f(t_1) \in \partial F(t_1)\), it holds that

\[
(t_1 - t_2)f(t_1) \geq F(t_1) - F(t_2) + \rho(t_1 - t_2)^2;
\]

6. For all \(\beta > 0\) and \(F_\beta(t) = F(\beta t) / \beta\), \(F_\beta(\cdot)\) is also a weakly convex sparseness measure and its corresponding parameters are \(\rho_\beta = \beta \rho\) and \(\alpha_\beta = \alpha\);
7. There exists a convex function \( G(\cdot) \) with subgradient \( g(\cdot) \) satisfying
\[
F(t) = \alpha |t| + G(t) + \rho t^2 \quad \text{and} \quad |g(t)| \leq -2\rho |t|.
\]

**Proof** The proof is postponed to Section 6.1. \( \blacksquare \)

According to Proposition 1, it can be implied that from the perspective of null space constant, if \( F(\cdot) \) is a sparseness measure, the performance of the non-convex problem (4) is at least as good as \( \ell_1 \)-minimization. The following theorem reveals that if \( F(\cdot) \) is further assumed weakly convex, the null space constant is the same as that of \( \ell_1 \)-minimization.

**Theorem 1** For any tuple \((J, A, K)\), the equality
\[
\gamma(J, A, K) = \gamma(\ell_1, A, K) \quad (11)
\]
holds.

**Proof** The proof is postponed to Section 6.2. \( \blacksquare \)

Though their null space constants are equal, it does not mean that for any fixed sparse signal \( x^* \), the performance of (4) and that of \( \ell_1 \)-minimization are the same. We will come back to this topic in Section 4.3.

### 4.1 The Scenario with Accurate Projection

The PGG method is theoretically analyzed in this subsection. Two lemmas are established for preparation. These lemmas are related with the optimization problem (4) and independent of specific recovery algorithms. Define \( \sigma_{\text{min}}(A) \) as the smallest nonzero singular value of \( A \).

**Lemma 2** For any tuple \((J, A, K)\) with \( \gamma(J, A, K) < 1 \) and positive constant \( M_0 \), there exists a constant \( C_1 \) such that
\[
\|x - x^*\|_2 \leq C_1 \|A(x - x^*)\|_2 \quad (12)
\]
for all vectors \( x^* \) and \( x \) satisfying \( \|x^*\|_0 \leq K, J(x) \leq J(x^*) \), and \( \|x - x^*\|_2 \leq M_0 \), and
\[
C_1 = \frac{1}{\sigma_{\text{min}}(A)} + \frac{M_0}{F(M_0)} \cdot \frac{\alpha \sqrt{N}(1 + \gamma(J, A, K))}{\sigma_{\text{min}}(A)(1 - \gamma(J, A, K))} \quad (13)
\]
is a finite constant.

**Proof** The proof is postponed to Section 6.3. \( \blacksquare \)

For the iterative solution \( x \) of the PGG method, since \( A(x - x^*) = e \), Lemma 2 declares that under some certain conditions, \( J \)-minimization is locally stable. It should be noticed that here \( C_1 \) grows without bound as \( M_0 \) approaches positive infinity. A better result has
been derived in [51, 52]. It reveals that if exact reconstruction condition (ERC) for tuple \((J, A, K)\) is satisfied (which can be guaranteed if \(\gamma(J, A, K) < 1\)), then robust reconstruction condition (RRC) holds with probability 1. In this setting, \(C_1\) is a finite number regardless of the distance between \(x\) and \(x^*\). However, since the algorithm starts with an initial solution (e.g., the least squares solution), Lemma 2 is sufficient in this paper. The result of Lemma 2 is in parallel with the one in [10], which declares that the optimization problem basis pursuit denoising (BPDN) is globally stable. In the noiseless scenario, i.e., \(e = 0\), Lemma 2 also implies that \(x^*\) is the only solution to \(J\)-minimization.

**Lemma 3** For any tuple \((J, A, K)\) with \(\gamma(J, A, K) < 1\), vector \(x^*\) with \(\|x^*\|_0 \leq K\), and positive constant \(M_0\), there exists a finite positive constant \(c\) such that

\[
J(x) - J(x^*) \geq c\|x - x^*\|_2^2
\]

for all vector \(x\) satisfying \(\|x - x^*\|_2 \leq M_0\) and \(\|x - x^*\|_2 \geq 2C_1\|A(x - x^*)\|_2\), where the constant \(c\) depends on the tuple \((J, A, K, x^*, M_0)\).

**Proof** The proof is postponed to Section 6.4.

The inequality (14) is somewhat similar to the concept of Lipschitz continuity, but with the difference that the inequality sign is reversed. According to (14), if the difference between \(J(x)\) and \(J(x^*)\) is small, \(x\) would not be far away from the desired sparse vector \(x^*\) as well. The existence of the positive constant \(c\) will play an important role in the theoretical convergence analysis. The following Theorem 2 demonstrates the main result on the local minima of \(J\)-minimization.

**Theorem 2** For any tuple \((J, A, K)\) with \(\gamma(J, A, K) < 1\), vector \(x^*\) with \(\|x^*\|_0 \leq K\), and positive constant \(M_0\), the inequality

\[
(x - x^*)^T \nabla J(x) > 0
\]

holds for all vector \(x\) and \(\eta \in (0, 1)\) satisfying

\[
\|x - x^*\|_2 \leq \min \left\{ M_0, \frac{\eta c}{-\rho} \right\},
\]

and \(\|x - x^*\|_2 \geq 2C_1\|A(x - x^*)\|_2\).

**Proof** The proof is postponed to Section 6.5.

Theorem 2 demonstrates the distribution of the local minima of \(J\)-minimization. As it reveals, for all local minima \(x\) in the area of (16), it should also satisfy

\[
\|x - x^*\|_2 \leq 2C_1\|A(x - x^*)\|_2.
\]
Recalling that $A(x - x^*) = e$, Theorem 2 implies that there would be no local minimum in the annulus it specified. Intuitively, if the initial solution satisfies (16), the PGG method would return a stably recovered solution.

The following Theorem 3 demonstrates the convergence property of PGG in one iteration. For simplicity, $x$ and $x^+$ represent $x(n)$ and $x(n+1)$, respectively.

**Theorem 3** For any tuple $(J, A, K)$ with $\gamma(J, A, K) < 1$, vector $x^*$ with $\|x^*\|_0 \leq K$, and positive constant $M_0$, if the previous iterative solution $x$ satisfies (16) and

$$\|x - x^*\|_2 \geq \frac{\mu \alpha^2 N}{2(1 - \eta)c} \kappa + \frac{C_2}{1 - \eta} \|A(x - x^*)\|_2$$

(17)

where $\eta \in (0, 1), \mu > 1$, and

$$C_2 = \max \left\{ 2C_1, \frac{\alpha \sqrt{N}}{\sigma_{\min}(A)c} \right\},$$

then the next iterative solution $x^+$ satisfies

$$\|x^+ - x^*\|_2^2 \leq \|x - x^*\|_2^2 - (\mu - 1) \alpha^2 N \kappa^2.$$  (18)

**Proof** The proof is postponed to Section 6.6.

According to Theorem 3, if the iterative solution $x(n)$ lies within a neighborhood of the desired sparse signal $x^*$ as (16), as long as the distance between $x(n)$ and $x^*$ is larger than a quantity linear in both the step size $\kappa$ and the noise term $\|e\|_2$, the next iterative solution $x(n+1)$ will definitely get closer to $x^*$, and the distance reduction is at least $(\mu - 1) \alpha^2 N \kappa^2$. Therefore, in finite iterations, the iterative solution $x(n)$ will get into the $(O(\kappa) + O(\|e\|_2))$-neighborhood of $x^*$. By choosing sufficiently small step size $\kappa$, the influence of the $O(\kappa)$ term can be omitted, and PGG returns a stably recovered solution.

The parameter $\mu$ is involved in Theorem 3, and it is just introduced in the theoretical analysis to characterize the convergence rate, as has been discussed in detail in Section III.E and Section III.F in [34]. By similar theoretical analysis as that in [34], we can derive a sequence which bounds the actual sequence in the convergence analysis and obtain its convergence rate.

To ensure that the PGG method converges, the sufficient condition (16) should be satisfied for the initial solution. We can simply choose $M_0$ and $\rho$ such that

$$M_0 = \|x(0) - x^*\|_2 < \frac{c}{-\rho}. \quad (19)$$

According to Lemma 3, the constant $c$ also depends on $\rho$, so we cannot derive the requirement of $\rho$ immediately. The following theorem reveals that weakly convex sparseness measures with appropriate $\rho$ will result in (19).
Theorem 4 For any tuple \((J, A, K)\) with \(\gamma(J, A, K) < 1\), vector \(x^*\) with \(\|x^*\|_0 \leq K\), and positive constant \(M_0\), consider a class of penalties

\[
J_{\beta}(x) = \sum_{i=1}^{N} F_{\beta}(x_i) = \frac{1}{\beta} J(\beta x), \quad \beta > 0.
\] (20)

Then there exists a positive constant \(\beta_1\) (depending on the tuple \((J, A, K, x^*, M_0)\)) such that for all \(\beta \in (0, \beta_1)\), the constraint (19) holds when penalty (20) is applied in \(J\)-minimization.

PROOF The proof is postponed to Section 6.7.

The reason why we adopt \(J(\beta x)/\beta\) instead of \(J(\beta x)\) lies in the fact that minimizing them subject to \(y = Ax\) returns the same optimal solution, and that with different \(\beta\), the parameter \(\rho_\beta\) of \(F_{\beta}(\cdot)\) varies while \(\alpha_\beta\) remains the same. This would allow us to focus on the influence of \(\rho\) which characterizes the non-convexity of the penalty functions. Define \(\rho_{\beta_1} = \beta_1 \rho\). According to Theorem 4, for penalty (20) with \(-\rho_\beta \in (0, -\rho_{\beta_1})\), the initial solution satisfies (19), and the convergence of PGG is guaranteed according to Theorem 3. The following proposition summaries the main result of the convergence performance of the PGG method.

Proposition 2 (Convergence of PGG). For any tuple \((J, A, K)\) with \(\gamma(J, A, K) < 1\), vector \(x^*\) with \(\|x^*\|_0 \leq K\), and positive constant \(M_0\), there exists a threshold \(\beta_1\) such that the non-convex optimization problem

\[
\begin{align*}
\text{argmin}_{x} & \quad \frac{1}{\beta} J(\beta x) \\
\text{subject to} & \quad y = Ax
\end{align*}
\] (21)

can be solved in using PGG with a recovered solution \(\hat{x}\) satisfying

\[
\|\hat{x} - x^*\|_2 \leq \frac{\alpha^2 N}{2c} + C_2\|e\|_2
\] (22)

provided that \(\|x(0) - x^*\|_2 \leq M_0\) and \(\beta \in (0, \beta_1)\).

The main contribution of this paper is the relationship between the initial criteria of the algorithm and the non-convexity of \(J\)-minimization, just as (19) reveals. An alternate interpretation is that the result is about the stability of \(\ell_1\)-minimization with respect to small perturbations of the \(\ell_1\) norm. To see this, the penalty function \(J_{\beta}(x)\) actually converges to a scaled version of the \(\ell_1\) norm uniformly over any bounded interval \([-T, T]\) as \(\beta\) approaches zero.

Now we can discuss the influence of the parameter \(\rho\) on the recovery performance. For \(\rho = 0\), \(F(\cdot) = |\cdot|\) and \(J(\cdot)\) is the \(\ell_1\) norm, and the performance of \(J\)-minimization is the same as that of \(\ell_1\)-minimization. The term \(c/(\rho)\) in (19) vanishes in this scenario. This is consistent in the fact that (4) is convex and any local minimum is identical to the global minimum. As \((\rho)\) increases, \(J\)-minimization is more non-convex, which might result in better...
recovery performance. But Proposition 2 implies that the non-convexity should be smaller than a threshold such that the algorithm converges from the initial solution. Therefore, one would expect that there exists \( \rho^* < 0 \) such that the performance of PGG improves as \((-\rho) \in (0, -\rho^*)\) increases, and degenerates rapidly as \((-\rho) \in (-\rho^*, +\infty)\) continues growing. As \((-\rho) \to 0_+\), the recovery performance tends to the case of \( J(\cdot) = \|\cdot\|_1 \). More theoretical results on the performance of \( J\)-minimization and its comparison with \( \ell_1 \)-minimization and \( \ell_0 \)-minimization are given in Section 4.3.

4.2 Extension to Approximate Projection and Compressible Signal

In this subsection, the performance of the APGG method is analyzed. Since \( A^\top B \) is adopted as the approximation of \( A^\dagger \), the iterative solution no longer satisfies \( Ax = y \). The following theorem gives the bound of \( \|A(x(n) - x^*)\|_2 \).

**Theorem 5** The iterative solution \( x(n) \) of APGG satisfies

\[
\|A(x(n) - x^*)\|_2 \leq \|e\|_2 + \|y\|_2 \zeta^{n+1} + \frac{1}{2} C_3(\zeta) \kappa,
\]

where \( C_3(\zeta) = 2 \alpha \sqrt{N} \|A\|_2 \zeta^{1/2} / (1 - \zeta) \).

**Proof** The proof is postponed to Section 6.8.

According to Theorem 5, if the accurate pseudo-inverse matrix is applied, i.e., \( \zeta = 0 \), the result is consistent in the scenario with accurate projection. For any fixed approximate precision \( \zeta \in (0, 1) \), as \( n \) approaches infinity and the step size \( \kappa \) is sufficiently small, the result reveals that the performance degradation caused by the approximate projection can be omitted. For the convenience of theoretical analysis, define a constant \( N_\kappa \) such that for all \( n \geq N_\kappa \),

\[
\|A(x(n) - x^*)\|_2 \leq \|e\|_2 + C_3(\zeta) \kappa.
\]

Since Lemma 2, Lemma 3, Theorem 2, and Theorem 4 in Section 4.1 are independent of specific algorithms, they still hold in this scenario with approximate projection. The following theorem demonstrates the convergence property of APGG in one iteration, which is a counterpart of Theorem 3.

**Theorem 6** For any tuple \((J, A, K)\) with \( \gamma(J, A, K) < 1 \), vector \( x^* \) with \( \|x^*\|_0 \leq K \), positive constant \( M_0 \), and \( A^\top B \) as approximate pseudo-inverse matrix with \( \zeta < 1 \), if the previous iterative solution \( x \) satisfies (16) and

\[
\|x - x^*\|_2 \geq \frac{\mu C_4(\zeta)}{1 - \eta} \kappa + \frac{C_5(\zeta)}{1 - \eta} \|e\|_2
\]

where \( \eta \in (0, 1) \), \( \mu > 1 \), \( C_4(\zeta) \) and \( C_5(\zeta) \) specified in the proof depend on the tuple \((J, A, K, x^*, M_0, A^\top B)\), then the next iterative solution \( x^+ \) satisfies

\[
\|x^+ - x^*\|_2^2 \leq \|x - x^*\|_2^2 - (\mu - 1) d \alpha^2 N \kappa^2
\]

where \( d = \|I - A^\top BA\|_2^2 \).
The proof is postponed to Section 6.9.

Similar to Theorem 3, Theorem 6 also reveals that if the initial solution $x(0)$ of APGG lies within a neighborhood of the desired sparse signal $x^*$ as (16), the iterative solution will get into the $(O(\kappa) + O(\|e\|_2))$-neighborhood of $x^*$ in finite iterations. This is somewhat interesting since the influence of approximate projection is only reflected on the coefficients other than an additional error term. In the noiseless scenario with sufficiently small step size $\kappa$, the sparse signal $x^*$ can be recovered with any given precision, even when a uniform approximate projection is applied in the method. Similar to Proposition 2, the following proposition summaries the main result of the convergence performance of the APGG method.

**Proposition 3 (Convergence of APGG).** For any tuple $(J, A, K)$ with $\gamma(J, A, K) < 1$, vector $x^*$ with $\|x^*\|_0 \leq K$, positive constant $M_0$, and $A^TB$ as approximate pseudo-inverse matrix with $\zeta < 1$, there exists a threshold $\beta_1$ such that the non-convex optimization problem

$$\arg\min_x \frac{1}{\beta} J(\beta x) \text{ subject to } y = Ax$$

can be solved in using APGG with recovered solution $\hat{x}$ satisfying

$$\|\hat{x} - x^*\|_2 \leq C_4(\zeta)\kappa + C_5(\zeta)\|e\|_2$$

provided that $\|x(0) - x^*\|_2 \leq M_0$ and $\beta \in (0, \beta_1)$.

In the main contributions, only the case of strictly sparse signal is analyzed and discussed. For compressible signal $x^*$, assume $\|x^* - x^*_T\|_2 \leq \tau$. It is easily calculated that

$$y = Ax^* + e = Ax^*_T + (e + A(x^* - x^*_T))$$

and

$$\|e + A(x^* - x^*_T)\|_2 \leq \|e\|_2 + \sigma_{\text{max}}(A)\tau,$$

where $\sigma_{\text{max}}(A)$ is the largest singular value of $A$. According to Proposition 3, the iterative solution $x(n)$ will get into the $(C_4(\zeta)\kappa + C_5(\zeta)(\|e\|_2 + \sigma_{\text{max}}(A)\tau))$-neighborhood of $x^*_T$. Since $x^*_T$ lies in the $\tau$-neighborhood of $x^*$, the distance between $x(n)$ and $x^*$ will be no more than

$$C_4(\zeta)\kappa + C_5(\zeta)\|e\|_2 + (C_5(\zeta)\sigma_{\text{max}}(A) + 1)\tau.$$ 

This reflects the performance degradation due to the noise and non-sparsity of the original signal.
4.3 Further Discussion on $J$-minimization

This work will be further solid if it is proved that as $(-\rho)$ approaches positive infinity, for fixed $x^*$, the performance of $J$-minimization would gradually improve from $\ell_1$-minimization to some problems with better performance, say $\ell_p$-minimization or even $\ell_0$-minimization. Here we will demonstrate some efforts towards this.

First, Proposition 1 reveals that from the perspective of null space constant, the performance of (4) lies between $\ell_1$-minimization and $\ell_0$-minimization. Notice that $\ell_p$ penalty ($0 < p < 1$) satisfies Definition 1.1)-3), therefore it is in accordance with the result that $\ell_p$ penalty leads to better performance than a convex one. Second, Theorem 1 reveals that the performance of $J$-minimization is the same as that of $\ell_1$-minimization, but we believe that this is not the case. Recalling the definition of null space constant, $\gamma(J, A, K)$ is independent of the particular choice of sparse signal $x^*$ and only depends on its sparsity $K$. Therefore it is not a tight quantity indicating the performance of $J$-minimization for tuple $(J, A, x^*)$.

**Proposition 4** Further assume that $F(\cdot)$ is bounded. For any tuple $(A, K)$ with $\gamma(\ell_0, A, K) < 1$ and vector $x^*$ with $\|x^*\|_0 \leq K$, the global optimum $\hat{x}^\beta$ of the problem

$$\arg\min_x \frac{1}{\beta} J(\beta x) \text{ subject to } Ax = Ax^*$$

satisfies $\lim_{\beta \to +\infty} \|\hat{x}^\beta - x^*\|_2 = 0$.

**Proof** The proof is postponed to Section 6.10.

As can be seen from Proposition 4, for some weakly convex sparseness measures (e.g., No. 6 in TABLE 1) and fixed sparse signal $x^*$, as $(-\rho)$ approaches positive infinity, the performance of $J$-minimization is identical to that of $\ell_0$-minimization. (One may notice that the condition in Proposition 4 is $\gamma(\ell_0, A, K) < 1$ rather than $\gamma(J, A, K) < 1$ or $\gamma(\ell_1, A, K) < 1$.) Therefore, for some sparse signals $x^*$, they cannot be recovered by $\ell_1$-minimization, but can be recovered by $J$-minimization. More generally, we conjecture that for fixed sparse signal $x^*$, as $(-\rho)$ increases from 0 to positive infinity, the performance of $J$-minimization would gradually improve from $\ell_1$-minimization to some problems with better performance, say $\ell_p$-minimization or even $\ell_0$-minimization. If this conjecture is true, the theoretical results would be in perfect accordance with the results in Fig. 2 in the following Section 5. We leave this as a future possible work and it is readdressed in the Conclusion.

5 Numerical Simulations

In this section, several simulations are performed to test the recovery performance of the PGG and APGG methods, and to verify the theoretical analysis. The sensing matrix $A$ is of size $M = 200$ and $N = 1000$, whose entries are independently and identically distributed
Figure 2: The figure shows the recovery performance of the PGG method with different sparsity-inducing penalties and different choices of $\rho$. The corresponding sparseness measures are from TABLE 1. The problem dimensions are $M = 200$ and $N = 1000$, and $K_{\text{max}}$ is the largest integer which guarantees 100% successful recovery.

Gaussian with zero mean and variance $1/M$. The locations of the non-zero entries of the sparse signal $x^*$ are randomly chosen among all possible choices, and these non-zero entries are independently Gaussian distributed with zero mean and the same variance. The sparse signal is finally normalized to have unit $\ell_2$ norm. In all simulations, the approximate $A^\dagger$ is calculated using the method introduced in Appendix A.

The first experiment tests the recovery performance of the PGG method in the noiseless scenario with different sparseness measures from TABLE 1. The No. 1 corresponds to the $\ell_1$ penalty. For each penalty with some certain $\rho$, the sparsity level $K$ varies from 1 to 100 with increment of one. If the recovery SNR (RSNR) is higher than 40dB, this recovery is regarded as a success. The simulation is repeated 100 trials to calculate the successful recovery probability versus sparsity $K$. Then the crucial sparsity $K_{\text{max}}$, which is the largest integer which guarantees 100% successful recovery, is recorded. The results are presented in Fig. 2. As is revealed, for the non-convex sparsity-inducing penalties, as $(−\rho)$ increases, the performance of PGG increases at first, and degenerates rapidly when $(−\rho)$ continues to grow. When $(−\rho)$ approaches zero, the performances of these penalties are close to that of the $\ell_1$ penalty. These results are consistent in the theoretical analysis and discussions of Proposition 2.

In the second experiment, the recovery performance of PGG and APGG is compared in the noiseless scenario with some typical sparse recovery algorithms, including OMP [13], the solution to BP [40], reweighted $\ell_1$ minimization [21], ISL0 [23], and IRLS [20]. In the simulation $K$ varies from 20 to 100. The PGG and APGG methods adopt the No. 6
Figure 3: The figure compares the successful recovery probability of different algorithms versus sparsity $K$ with $M = 200$ and $N = 1000$. The approximate precision of approximate $A^\dagger$ is $\zeta = 0.91$.

sparseness measure in TABLE 1 with $\sigma = 10$, and the step size is set to $10^{-6}$. The iterative number for calculating inexact pseudo-inverse matrices is 0, which means that $\zeta A^T$ is adopted. The approximate precision of approximate $A^\dagger$ is $\zeta = 0.91$. The simulation is repeated 200 trials to calculate the successful recovery probability versus sparsity $K$. The simulation results are demonstrated in Fig. 3. As can be seen, IRLS, PGG, and APGG guarantee successful recovery for larger sparsity $K$ than the other reference algorithms. It also reveals that in the noiseless scenario with sufficiently small step size, the approximate projection has little influence on the recovery performance of APGG.

In the last experiment, the recovery precisions of the PGG and APGG methods are simulated under different settings of measurement noise and step size. In the simulation, the sparsity level $K = 30$. The same sparseness measure as that in the previous experiment is adopted, and the iterative number for calculating approximate $A^\dagger$ is 4 such that $\zeta = 0.22$. The simulation is repeated 100 trials to calculate the mean squared error (MSE), and the results are shown in Fig. 4. As can be seen, there is almost no difference between the performance of APGG and that of PGG. In the noisy scenario, the recovery SNR (RSNR) is dependent on both the measurement SNR (MSNR) and the step size. For fixed MSNR, as the step size decreases, the RSNR improves at first, and remains the same when the step size is sufficiently small. Larger MSNR results in larger RSNR limit. In the noiseless scenario, the RSNR improves as the step size decreases, and it can be arbitrarily large by adopting sufficiently small step size. These results are accordant with Proposition 2 and Proposition 3, which implies that the recovery error is linear in both the noise term and the step size.
Figure 4: The figure demonstrates the recovery precision of the PGG and APGG methods under different measurement noise and step size with $M = 200$, $N = 1000$, and $K = 30$. The approximate precision of approximate $A^\dagger$ is $\zeta = 0.22$.

6 Proofs

6.1 Proof of Lemma 1

PROOF 1) Consider the non-trivial scenario that $t_1$ and $t_2$ are both non-zero. Since $F(t)/t$ is non-increasing on $(0, +\infty)$, it is easily checked that

$$F(t_1) = F(|t_1|) \geq \frac{|t_1|F(|t_1| + |t_2|)}{|t_1| + |t_2|};$$

$$F(t_2) = F(|t_2|) \geq \frac{|t_2|F(|t_1| + |t_2|)}{|t_1| + |t_2|}.$$

Summing these two inequalities, together with the non-decreasing property of $F(\cdot)$ on $[0, +\infty)$, it holds that

$$F(t_1) + F(t_2) \geq F(|t_1| + |t_2|) \geq F(|t_1 + t_2|) = F(t_1 + t_2).$$

2) Since $F(\cdot)$ is non-decreasing on $[0, +\infty)$, the directional derivative [29]

$$D_F(t, -1) = \lim_{\theta \to 0^+} \frac{F(t - \theta) - F(t)}{\theta} \leq 0$$

for all $t > 0$. Thus the definition of the generalized gradient set [29] implies that for all $f(t) \in \partial F(t)$, $f(t) \geq 0$.

3) The continuity of $F(\cdot)$ can be easily checked by Proposition 4.3 in [29], i.e., there exists a convex function $H(\cdot)$ such that $F(\cdot) = H(\cdot) + \rho \| \cdot \|_2^2$, and the continuity of convex functions.
As for the inequality, we only need to consider $t > 0$. Since $F(t)/t$ is non-increasing on $(0, +\infty)$ and

$$\lim_{t \to 0^+} \frac{F(t)}{t} = \lim_{t \to 0^+} \left( \frac{H(t)}{t} + \rho t \right) = \lim_{t \to 0^+} \frac{H(t) - H(0)}{t} = \alpha$$

is bounded, it holds that for all $t > 0$, $F(t)/t \leq \alpha$.

4) It is easy to check that $F(\cdot)$ is also weakly convex with parameter $\rho$ on $(-\infty, 0]$ and that for all $t \in \mathbb{R}$, $\partial F(-t) = -\partial F(t)$. Thus we only need to consider the case of $t > 0$. Due to the non-increasing property of $F(t)/t$, it can be verified that

$$\frac{F(t+\theta) - F(t)}{\theta} \leq \frac{F(t)}{t}$$

holds for all $\theta > 0$. Therefore the definition of the generalized gradient implies

$$0 \leq f(t) \leq \lim_{\theta \to 0^+} \frac{F(t-\theta) - F(t)}{\theta} \leq \frac{F(t)}{t} \leq \alpha.$$

5) First, if $(t_1, t_2)$ satisfies the inequality (10), $(-t_1, -t_2)$ also satisfies it, thus we only need to consider the scenario that $t_1 \geq 0$.

If $t_1 = 0$, the result is obvious since $\rho \leq 0$. If $t_1 > 0$ and $t_2 \geq 0$, according to Proposition 4.8 in [29] and the fact that $F(\cdot)$ is weakly convex with parameter $\rho$ on $[0, +\infty)$, the inequality (10) is still obvious. If $t_2 < 0$, then $-t_2 > 0$. Since $f(t_1) \geq 0$, it can be derived that

$$(t_1 - t_2)f(t_1) \geq (t_1 - (-t_2))f(t_1) \geq F(t_1) - F(-t_2) + \rho(t_1 + t_2)^2 \geq F(t_1) - F(t_2) + \rho(t_1 - t_2)^2.$$

To sum up, the inequality (10) is proved.

6) It is easy to check that $F_\beta(\cdot)$ satisfies Definition 1.1)-3). Since $F_\beta(t) = H(\beta t)/\beta + \beta \rho t^2$ and $H_\beta(t) = H(\beta t)/\beta$ is convex, the parameter $\rho_\beta = \beta \rho$. In addition, since

$$\frac{F_\beta(t)}{t} = \frac{H(\beta t)}{\beta t} + \beta \rho t,$$

the inequality $F_\beta(t) \leq \alpha|t|$ holds as well. Thus $\alpha_\beta$ remains the same as $\alpha$.

7) Assume $F(t) = H(t) + \rho t^2$ and decompose $H(\cdot)$ by $H(t) = \alpha|t| + G(t)$. Since $H(\cdot)$ is convex, according to the definition of $\alpha$, $G(\cdot)$ is convex as well. In addition, since $|f(t)| \leq \alpha$, it can be easily derived that $|g(t)| \leq -2\rho|t|$.

6.2 Proof of Theorem 1

PROOF Define a class of penalties

$$J_\beta(x) = \sum_{i=1}^N F_\beta(x_i) = \frac{1}{\beta} J(\beta x), \quad \beta > 0.$$
We first prove that for all $\beta > 0$,

$$\gamma(J, A, K) = \gamma(J_{\beta}, A, K).$$

This can be simply proved from the definition of the null space constant

$$\gamma(J, A, K) = \sup_{0 \neq z \in \mathcal{N}(A), \#S \leq K} \frac{J(z_S)}{J(z_{S^c})}$$

and the fact that for all $\beta > 0$, $\beta z \in \mathcal{N}(A)$ is equivalent to $z \in \mathcal{N}(A)$.

Now we can prove $\gamma(J, A, K) = \gamma(\ell_1, A, K)$. If not, then there exists $\delta > 0$ such that for all $\beta > 0$,

$$\gamma(J_{\beta}, A, K) \leq \gamma(\ell_1, A, K) - \delta. \quad (27)$$

According to the definition of the null space constant, there exist $z \in \mathcal{N}(A)$ and set $S$ with $\#S \leq K$ such that

$$\frac{\|z_S\|_1}{\|z_{S^c}\|_1} \geq \gamma(\ell_1, A, K) - \delta/3. \quad (28)$$

In addition, since for fixed $z$ and $S$,

$$\lim_{\beta \to 0^+} \frac{J_{\beta}(z_S)}{J_{\beta}(z_{S^c})} = \frac{\|z_S\|_1}{\|z_{S^c}\|_1},$$

there exists $\beta_0 > 0$ such that for all $0 < \beta \leq \beta_0$,

$$\frac{J_{\beta}(z_S)}{J_{\beta}(z_{S^c})} \geq \frac{\|z_S\|_1}{\|z_{S^c}\|_1} - \delta/3. \quad (29)$$

Combining (28) with (29), it can be derived that

$$\frac{J_{\beta}(z_S)}{J_{\beta}(z_{S^c})} \geq \gamma(\ell_1, A, K) - 2\delta/3, \quad (30)$$

which contradicts (27).

6.3 Proof of Lemma 2

Proof Define $u = x - x^*$ and decompose $u$ by $u = z + z^\perp$, where $z \in \mathcal{N}(A)$ and $z^\perp \in \mathcal{N}^\perp(A)$ which denotes the orthogonal complement of $\mathcal{N}(A)$. Therefore $A z^\perp = A u$. Since $\sigma_{\min}(A)$ is the smallest nonzero singular value of $A$,

$$\|z^\perp\|_2 \leq \frac{1}{\sigma_{\min}(A)} \|A u\|_2. \quad (31)$$

Now consider the upper bound of $\|z\|_2$. Recalling that $x^*$ is supported on $T$, it can be derived that

$$J(x) = J(x^* + u) = J(x^* + u_T) + J(u_{T^c}). \quad (32)$$
According to Lemma 1.1), (32) and the assumption that \( J(x) \leq J(x^*) \), it can be derived that
\[
J(u^Tc) \leq J(x^*) - J(x^* + u_T) \leq J(u_T). \tag{33}
\]
By the decomposition of \( u \), it can be derived from Lemma 1.1) that
\[
J(u^Tc) = J(z^Tc + z^\perp) \geq J(z^Tc) - J(z^\perp),
\]
\[
J(u_T) = J(z_T + z^\perp) \leq J(z_T) + J(z^\perp).
\]
Together with (33),
\[
J(z^Tc) - J(z_T) \leq J(z^\perp). \tag{34}
\]
On the other hand, according to the definition of null space constant (6),
\[
J(z^Tc) - J(z_T) = J(z) - 2J(z_T)
\geq J(z) - 2 \frac{\gamma(J, A, K)}{1 + \gamma(J, A, K)} J(z)
= 1 - \frac{\gamma(J, A, K)}{1 + \gamma(J, A, K)} J(z). \tag{35}
\]
(34) and (35) together imply
\[
J(z) \leq \frac{1 + \gamma(J, A, K)}{1 - \gamma(J, A, K)} J(z^\perp). \tag{36}
\]
According to Lemma 1.3) and (31),
\[
J(z^\perp) \leq \alpha \|z^\perp\|_1 \leq \alpha \sqrt{N} \|z^\perp\|_2 \leq \frac{\alpha \sqrt{N}}{\sigma_{\min}(A)} \|Au\|_2. \tag{37}
\]
Since for \( 1 \leq i \leq N \), \( |z_i| \leq \|z\|_2 \leq \|u\|_2 \leq M_0 \), it can be calculated that
\[
J(z) = \sum_{i=1}^{N} F(z_i) \geq \sum_{i=1}^{N} \frac{F(M_0)}{M_0} |z_i| \geq \frac{F(M_0)}{M_0} \|z\|_2, \tag{38}
\]
where the first inequality is due to Definition 1.3). Thus (36), (37), and (38) imply
\[
\|z\|_2 \leq \frac{M_0}{F(M_0)} \cdot \frac{\alpha \sqrt{N}(1 + \gamma(J, A, K))}{\sigma_{\min}(A)(1 - \gamma(J, A, K))} \|Au\|_2. \tag{39}
\]
To sum up, since \( \|u\|_2 \leq \|z\|_2 + \|z^\perp\|_2 \) with (31) and (39), there exists a finite constant \( C_1 \) such that (12) holds, and it is given explicitly by (13).

\[\blacksquare\]
6.4 Proof of Lemma 3

**Proof** First of all, consider vector \( x \) that further satisfies \( A(x - x^*) = 0 \), which implies that \( x - x^* = z \in N(A) \). Therefore

\[
J(x) - J(x^*) = J(x^* + z) - J(x^*)
= J(z_Tc) - (J(x^*) - J(x^* + z_T))
\geq J(z_Tc) - J(z_T)
\geq \frac{1 - \gamma(J, A, K)}{1 + \gamma(J, A, K)} J(z),
\]

where the last inequality is due to (35). According to (38),

\[
J(x) - J(x^*) \geq F(M_0) \cdot \frac{1 - \gamma(J, A, K)}{1 + \gamma(J, A, K)} \|z\|_2.
\]

(40)

Now turn to the scenario that \( A(x - x^*) \neq 0 \). According to Lemma 2, \( J(x) > J(x^*) \).

Define

\[
r(x) = \frac{J(x) - J(x^*)}{\|x - x^*\|_2}
\]

and \( c = \inf_{x \in S} r(x) \) with

\[
S = \{x : A(x - x^*) \neq 0, 2C_1 \|A(x - x^*)\|_2 \leq \|x - x^*\|_2 \leq M_0\}.
\]

Since \( r(x) > 0 \) for all \( x \in S, c \geq 0 \). Now we prove \( c > 0 \).

Otherwise, assume \( c = 0 \). Then there exists a sequence \( \{x_i\} \in S \) such that for all \( \varepsilon > 0 \), there exists \( N_\varepsilon > 0 \) and for all \( i > N_\varepsilon \), \( r(x_i) \leq \varepsilon \). Since the sequence \( \{x_i\} \) is bounded, according to Bolzano-Weierstrass Theorem, there exists a subsequence \( \{x_{n_i}\} \) which converges to some point, say, \( x_0 \). Notice that \( x_0 \) belongs to the closure of \( S \), i.e., \( \overline{S} \). Since \( r(\cdot) \) is continuous on \( \overline{S} \), the sequence \( r(x_{n_i}) \) converges and its limit is \( r(x_0) = 0 \). This contradicts (40) and the fact that for all \( x \in S \), \( r(x) > 0 \).

To sum up, Lemma 3 is proved.

6.5 Proof of Theorem 2

**Proof** According to Lemma 1.5), it can be derived that

\[
(x - x^*)^T \nabla J(x) = \sum_{i=1}^{N} (x_i - x_i^*) f(x_i)
\geq \sum_{i=1}^{N} (F(x_i) - F(x_i^*) + \rho(x_i - x_i^*)^2)
= J(x) - J(x^*) + \rho \|x - x^*\|_2^2.
\]

(42)
Since \( \|x - x^*\|_2 \leq \frac{pc}{\rho} \), Lemma 3 and (42) imply
\[
(x - x^*)^T \nabla J(x) \geq c\|x - x^*\|_2 + \rho\|x - x^*\|_2^2 \\
\geq (1 - \eta)c\|x - x^*\|_2, \tag{43}
\]
which completes the proof.

6.6 Proof of Theorem 3

**Proof** Define \( u = x - x^* \) and \( u^+ = x^+ - x^* \). According to (8) and (9), it can be derived that
\[
u^+ = u - \kappa(I - A^\dagger A)\nabla J(x),
\]
which further implies
\[
\|u^+\|_2^2 = \|u\|_2^2 + \kappa^2\|(I - A^\dagger A)\nabla J(x)\|_2^2 - 2\kappa u^T(I - A^\dagger A)\nabla J(x). \tag{44}
\]
According to Lemma 1.4), the second item on the right side of (44) can be bounded as
\[
\|(I - A^\dagger A)\nabla J(x)\|_2^2 \leq \|\nabla J(x)\|_2^2 \leq \alpha^2 N.
\]
The third item on the right side of (44) can be decomposed to
\[
u^T(I - A^\dagger A)\nabla J(x) = u^T\nabla J(x) - u^T A^\dagger A \nabla J(x).
\]
On one hand, according to the proof of Theorem 2, (43) implies that
\[
u^T\nabla J(x) \geq (1 - \eta)c\|u\|_2.
\]
On the other hand,
\[
u^T A^\dagger A \nabla J(x) \leq \|A^\dagger\|_2 \|Au\|_2 \|\nabla J(x)\|_2 \\
\leq \frac{\alpha\sqrt{N}}{\sigma_{\min}(A)} \|Au\|_2.
\]
Substituting these inequalities into (44), the right side of (44) can be bounded as
\[
\|u\|_2^2 + \alpha^2 N \kappa^2 - 2(1 - \eta)c\|u\|_2 \kappa + \frac{2\alpha\sqrt{N}}{\sigma_{\min}(A)} \|Au\|_2 \kappa \leq \|u\|_2^2 - (\mu - 1)\alpha^2 N \kappa^2,
\]
where the last inequality is due to (17).  \( \blacksquare \)
6.7 Proof of Theorem 4

Proof According to Lemma 1.6), the parameters of $F_\beta(\cdot)$ are $\rho_\beta = \beta \rho$ and $\alpha_\beta = \alpha$. Lemma 1.7) implies that $F_\beta(\cdot)$ can be decomposed to

$$F_\beta(t) = \alpha |t| + G_\beta(t) + \beta \rho t^2,$$

where $G_\beta(\cdot)$ is convex and its subgradient satisfies $|g_\beta(t)| \leq -2\beta \rho |t|$. Therefore

$$J_\beta(x) - J_\beta(x^*) = \alpha(\|x\|_1 - \|x^*\|_1) + \beta \rho(\|x\|_2^2 - \|x^*\|_2^2) + \sum_{i=1}^{N} (G_\beta(x_i) - G_\beta(x^*_i)). \quad (45)$$

According to Theorem 1, $\gamma(\ell_1, A, K) = \gamma(J, A, K) < 1$, therefore Lemma 3 implies that there exists a positive constant $c_0$ such that

$$\alpha(\|x\|_1 - \|x^*\|_1) \geq c_0 \|x - x^*\|_2$$

and $c_0$ is independent of $\rho$. In addition,

$$\|x\|_2^2 - \|x^*\|_2^2 = (\|x\|_2 + \|x^*\|_2)(\|x\|_2 - \|x^*\|_2) \leq (\|x\|_2 + \|x^*\|_2)\|x - x^*\|_2.$$

Furthermore, since $G_\beta(\cdot)$ is convex,

$$\sum_{i=1}^{N} (G_\beta(x_i) - G_\beta(x^*_i)) \geq \sum_{i=1}^{N} g_\beta(x^*_i)(x_i - x^*_i) \geq 2\beta \rho \sum_{i=1}^{N} |x^*_i||x_i - x^*_i| \geq 2\beta \rho \|x^*\|_2 \|x - x^*\|_2,$$

where the last inequality is due to Cauchy-Schwartz inequality.

With the above inequalities and (45), it holds that

$$J_\beta(x) - J_\beta(x^*) \geq (c_0 + \beta \rho(\|x\|_2 + 3\|x^*\|_2))\|x - x^*\|_2.$$

Thus the constant $c_\beta$ of $J_\beta(\cdot)$ satisfies

$$c_\beta \geq c_0 + \beta \rho(\|x\|_2 + 3\|x^*\|_2) \geq c_0 + \beta \rho(M_0 + 4\|x^*\|_2),$$

where the last inequality is due to

$$\|x\|_2 \leq \|x - x^*\|_2 + \|x^*\|_2 \leq M_0 + \|x^*\|_2.$$

Therefore,

$$\frac{c_\beta}{-\rho_\beta} = \frac{c_\beta}{-\beta \rho} \geq \frac{c_0}{-\beta \rho} - (M_0 + 4\|x^*\|_2).$$

Since $M_0 = \|x(0) - x^*\|_2$ is a bounded number, there exists a positive constant $\beta_1$ such that for all $\beta \in (0, \beta_1),$

$$M_0 < \frac{c_0}{-\beta \rho} - (M_0 + 4\|x^*\|_2) \leq \frac{c_\beta}{-\rho_\beta},$$

which completes the proof.\[\]
6.8 Proof of Theorem 5

PROOF First, we prove that
\[ \|y - Ax(n)\|_2 \leq \|y\|_2 \zeta^{n+1} + \frac{1}{2} C_3(\zeta) \kappa. \]  
(46)

For \( n = 0 \), the initialization is \( x(0) = A^T By \), which satisfies
\[ \|y - Ax(0)\|_2 = \|y - AA^T By\|_2 \leq \|y\|_2 \zeta. \]  
(47)

For the \((n + 1)\)th iteration, the iterative solution obeys
\[ x(n + 1) = A^T By + (I - A^T BA)(x(n) - \kappa \nabla J(x(n))), \]  
(48)

which satisfies
\[ \|y - Ax(n + 1)\|_2 = \|(I - AA^T B)(y - A(x(n) - \kappa \nabla J(x(n))))\|_2 \]
\[ \leq \|y - Ax(n) + \kappa A \nabla J(x(n))\|_2 \zeta \]
\[ \leq \|y - Ax(n)\|_2 \zeta + \|A\|_2 \|\nabla J(x(n))\|_2 \zeta \kappa \]
\[ \leq \|y - Ax(n)\|_2 \zeta + \alpha \sqrt{N} \|A\|_2 \zeta \kappa. \]

Together with (47), it can be derived by recursion that
\[ \|y - Ax(n)\|_2 \leq \|y - Ax(0)\|_2 \zeta^n + \frac{\alpha \sqrt{N} \|A\|_2 \zeta}{1 - \zeta} \cdot \kappa \]
\[ \leq \|y\|_2 \zeta^{n+1} + \frac{1}{2} C_3(\zeta) \kappa, \]

Now we turn to the proof of Theorem 5. Since \( y = Ax^* + e \), it can be derived that
\[ \|A(x(n) - x^*)\|_2 \leq \|y - Ax^*\|_2 + \|y - Ax(n)\|_2 \]
\[ \leq \|e\|_2 + \|y\|_2 \zeta^{n+1} + \frac{1}{2} C_3(\zeta) \kappa, \]

which completes the proof.

\[ \blacksquare \]

6.9 Proof of Theorem 6

PROOF Similar to the proof of Theorem 3, define \( u = x - x^* \) and \( u^+ = x^+ - x^* \). According to (48), it can be derived that
\[ u^+ = u + A^T B(y - Ax) - \kappa (I - A^T BA) \nabla J(x), \]

which further implies
\[ \|u^+\|_2^2 = \|u\|_2^2 + \|A^T B(y - Ax)\|_2^2 + 2u^T A^T B(y - Ax) \]
\[ + \kappa^2 \| (I - A^T BA) \nabla J(x) \|_2^2 \]
\[ - 2\kappa u^T (I - A^T BA) \nabla J(x) \]
\[ - 2\kappa (y - Ax)^T B^T A (I - A^T BA) \nabla J(x). \]  
(49)
According to Theorem 5 and its proof, for the second item on the right side of (49),

\[ \| A^T B (y - Ax) \|_2^2 \leq \| B^T A A^T B \|_2 \| y - Ax \|_2^2 \leq (1 + \zeta) \| B \|_2 C_3^2(\zeta) \kappa^2. \]

For the third item,

\[ u^T A^T B (y - Ax) = (Au)^T B (y - Ax) \leq \| B \|_2 C_3(\zeta) \kappa \left( \| e \|_2 + C_3(\zeta) \kappa \right). \]

For the forth item,

\[ \| (I - A^T B A) \nabla J(x) \|_2^2 \leq \| I - A^T B A \|_2^2 \alpha^2 N = d \alpha^2 N \]

For the fifth item, it can be decomposed to

\[ u^T (I - A^T B A) \nabla J(x) = u^T \nabla J(x) - u^T A^T B A \nabla J(x). \]

Again, according to the proof of Theorem 2, if

\[ \| x - x^* \|_2 \geq 2C_1(\| e \|_2 + C_3(\zeta) \kappa), \]

(43) implies that

\[ u^T \nabla J(x) \geq (1 - \eta) c \| u \|_2, \]

and

\[ u^T A^T B A \nabla J(x) = (Au)^T B A \nabla J(x) \leq \alpha \sqrt{N} \| A \|_2 \| B \|_2 (\| e \|_2 + C_3(\zeta) \kappa). \]

For the last item,

\[ (y - Ax)^T B^T A (I - A^T B A) \nabla J(x) = (y - Ax)^T B^T (I - A A^T B) A \nabla J(x) \geq - \alpha \sqrt{N} \| A \|_2 \| B \|_2 C_3(\zeta) \kappa. \]

Together with the above inequalities, (49) can be simplified to

\[ \| u^+ \|_2^2 \leq \| u \|_2^2 + d \alpha^2 N \kappa^2 - 2c (1 - \eta) \| u \|_2 - C_0(\zeta) \kappa - C_7(\zeta) \| e \|_2 \kappa, \]

where

\[ C_6(\zeta) = \frac{\| B \|_2 C_3(\zeta)}{2c} \left( 2(1 + \zeta) \alpha \sqrt{N} \| A \|_2 + (3 + \zeta) C_3(\zeta) \right), \]

\[ C_7(\zeta) = \frac{\| B \|_2}{c} \left( \alpha \sqrt{N} \| A \|_2 + C_3(\zeta) \right). \]
Define the constants
\[ C_4(\zeta) = \max \left\{ 2C_1C_3(\zeta), \frac{d\alpha^2N}{2c} + C_6(\zeta) \right\}, \]
\[ C_5(\zeta) = \max \{ 2C_1, C_7(\zeta) \}. \]

Under the assumption (24), inequality (51) implies
\[
\|u^+\|_2^2 \leq \|u\|_2^2 + d\alpha^2N\kappa^2 - \mu d\alpha^2N\kappa^2
= \|u\|_2^2 - (\mu - 1)d\alpha^2N\kappa^2,
\]
which arrives the final inequality. \( \blacksquare \)

### 6.10 Proof of Proposition 4

**Proof** It is easy to check that the optimization problem is equivalent to the following
\[ \arg\min_x J(\beta x) \text{ subject to } Ax = Ax^*. \]

According to the definition of null space constant and \( \gamma(\ell_0, A, K) < 1 \), for all nonzero vector \( z \in N(A) \), \( z \) has at least \((2K + 1)\) nonzero entries. Therefore, any \((2K + 1)\) column vectors of \( A \) are independent. Since \( F(\cdot) \) is non-decreasing and bounded on \([0, +\infty)\), without loss of generality, we assume \( \lim_{t \to +\infty} F(t) = C > 0 \).

For all \( \varepsilon > 0 \), define
\[ \delta = \frac{\varepsilon}{\sqrt{N(D\sigma_{\max}(A) + 1)}} > 0 \]
where \( D^{-1} \) is the smallest singular value of all \((2K + 1)\) column submatrices of \( A \) \((D^{-1} \) is nonzero since any \((2K + 1)\) column vectors of \( A \) are independent). Since \( F(\cdot) \) is non-decreasing on \([0, +\infty)\), there exists a constant \( \beta_0 > 0 \) such that for all \( \beta > \beta_0 \) and for all \( t > \delta \), \( F(\beta t) \geq \frac{K}{K + 1}C \). We now prove that for all \( \beta > \beta_0 \), \( \hat{x}^\beta \) has at most \((K + 1)\) entries with absolute value no less than \( \delta \). This is due to the fact that (define \( I_\beta \) as the set of index \( i \) satisfying \(|\hat{x}_i^\beta| \geq \delta\))
\[
KC \geq J(\beta x^*) \geq J(\beta \hat{x}^\beta) \geq \sum_{i \in I_\beta} F(\beta \hat{x}_i^\beta) \geq \frac{K}{K + 1}C \cdot |I_\beta|
\]
which implies \(#I_\beta \leq K + 1\). Together with \( K\)-sparse signal \( x^* \), at most \((2K + 1)\) entries of \( \hat{x}^\beta - x^* \) are with absolute value no less than \( \delta \).

Now we prove that for all \( \beta > \beta_0 \), \( \|\hat{x}^\beta - x^*\|_2 \leq \varepsilon \). Define \( z^\beta = \hat{x}^\beta - x^* \) and \( I^\beta \) as the set of index \( i \) satisfying \(|z_i^\beta| \geq \delta\), then as has been proved, \(#I^\beta \leq 2K + 1\). On one hand,
\[
\|z_{(I^\beta)c}\|_2 \leq \sqrt{N}\delta.
\]
On the other hand, since \( |I^\beta| \leq 2K + 1 \) and \( Az^\beta = 0 \),
\[
\|z^\beta_{I^\beta}\|_2 \leq D\|Az^\beta_{I^\beta}\|_2 \leq D\sigma_{\text{max}}(A)\|z^\beta_{(I^\beta)^c}\|_2 \leq D\sigma_{\text{max}}(A)\sqrt{N}\delta.
\]
Therefore,
\[
\|z^\beta\|_2 \leq \|z^\beta_{I^\beta}\|_2 + \|z^\beta_{(I^\beta)^c}\|_2 \leq (D\sigma_{\text{max}}(A) + 1)\sqrt{N}\delta = \varepsilon.
\]
To sum up, we have proved that for all \( \varepsilon > 0 \), there exists \( \beta_0 > 0 \) such that for all \( \beta > \beta_0 \), \( \|\hat{x}^\beta - x^*\|_2 \leq \varepsilon \). This directly leads to Proposition 4.

7 Conclusion

This paper considers the convergence guarantee of non-convex optimization problems for sparse recovery. A class of weakly convex sparseness measures is adopted to constitute the sparsity-inducing penalties. The distribution of local minima in a neighborhood of the sparse signal (with radius in inverse proportion to the non-convexity of penalties) reveals that as long as the non-convexity is below a threshold, all local minima in this neighborhood can be regarded as stable solutions. The convergence analysis of the PGG and APGG methods reveals that when the non-convexity is below a threshold, the recovery error will be linear in both the noise term and the step size. As for the APGG method, the influence of approximate projection is reflected in the coefficients rather than an additional error term. Therefore, in the noiseless scenario with sufficiently small step size, APGG derives a solution with any given precision. Some initiative result on \( J \)-minimization reveals that its performance might be better than the convex \( \ell_1 \)-minimization. Simulation results verify the theoretical analysis in this paper, and the recovery performance of APGG is not much influenced by the approximate projection.

There are several future directions to be explored. The first direction is to derive a more exact expression of the threshold \( \rho^* \), which would be very helpful for parameter selection rules in non-convex algorithms to solve practical sparse recovery problems. The second direction is to study the performance of \( J \)-minimization for tuple \( (J, A, x^*) \). In this paper we mainly utilize the null space constant to characterize its performance, and it is tight for tuple \( (J, A, K) \). But for fixed sparse signal \( x^* \), as \( -\rho \) increases, the performance of \( J \)-minimization should be different, as is revealed in the simulation with Fig. 2. The third possible direction is to improve the performance of sparse recovery by solving a sequence of optimization problems with different \( \rho \). The major concern would be the selection rules of the sequence of \( \rho \) such that the recovered solution for the previous \( \rho \) would lie in the convergence neighborhood for the next \( \rho \).

A Approximate Calculation of \( A^\dagger \)

The methods of computing \( A^\dagger \) have been developed to a mature technology. They are roughly classified into two categories: direct methods [44] and iterative methods [45]. Di-
rect methods are mainly based on matrix decompositions, such as QR decomposition [44] and singular value decomposition [46, 47]. Iterative methods, on the other hand, derive the pseudo-inverse matrix iteratively. To develop more accurate solutions, they cost more computational resources. Therefore, the iterative methods are preferred if approximate pseudo-inverse matrix can be applied to reduce the computational complexity.

A well-known iterative method introduced by Ben-Israel et al. [48] is

\[
Y_0 = \zeta A^T, \\
Y_k = Y_{k-1}(2I - AY_{k-1})
\]

with the parameter \( \zeta \) satisfying \( 0 < \zeta < 2/\|AA^T\|_1 \), where \( \| \cdot \|_1 \) returns the maximum absolute column sum of the matrix. Simple calculation derives that

\[
\|I - AY_0\|_2 = \|I - \zeta AA^T\|_2 < 1, \\
\|I - AY_k\|_2 \leq \|I - AY_{k-1}\|_2^2 \leq \|I - AY_0\|_2^k,
\]

which means that the method is quadratic convergence.

In this paper, it is assumed that the approximate pseudo-inverse matrix is of the form \( A^TB \), i.e., the transpose of \( A \) multiplied by a matrix \( B \in \mathbb{R}^{M \times M} \). \( B \) can be considered as the approximation of \((AA^T)^{-1}\). It can be verified that many, if not all, iterative methods satisfy this assumption [45, 48, 49].

References


