

Theory of Spectral Method for Union of Subspaces-Based Random Geometry Graph

Gen Li and Yuantao Gu*

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Abstract

Spectral Method is a commonly used scheme to cluster data points lying close to Union of Subspaces by first constructing a Random Geometry Graph, called Subspace Clustering. This paper establishes a theory to analyze this method. Based on this theory, we demonstrate the efficiency of Subspace Clustering in fairly broad conditions. The insights and analysis techniques developed in this paper might also have implications for other random graph problems. Numerical experiments demonstrate the effectiveness of our theoretical study.

Keywords: Spectral Method, Union of Subspaces, Subspace Clustering, Random Graph, Random Geometry Graph

1 Introduction

1.1 Motivation

Union of Subspaces (UoS) model serves as an important model in statistical machine learning. Briefly speaking, UoS models those high-dimensional data, encountered in many real-world problems, which lie close to low-dimensional subspaces corresponding to several classes to which the data belong, such as hand-written digits (Hastie and Simard, 1998), face images (Basri and Jacobs, 2003), DNA microarray data (Parvaresh et al., 2008), and hyperspectral images (Chen et al., 2011), to name just a few. A fundamental task in processing data points in UoS is to cluster these data points, which is known as Subspace Clustering (SC). Applications of SC has spanned all over science and engineering, including motion segmentation (Costeira and Kanade, 1998; Kanatani, 2001), face recognition (Wright et al., 2008), and classification of diseases (McWilliams and Montana, 2014) and so on. We refer the reader to the tutorial paper (Vidal, 2011) for a review of the development of SC.

*The authors are with Department of Electronic Engineering, Tsinghua University, Beijing 100084, China. The corresponding author of this paper is Y. Gu (gyt@tsinghua.edu.cn).

Considering the wide applications of SC, numerous algorithms have been developed for subspace clustering (Tipping and Bishop, 1999; Tseng, 2000; Vidal et al., 2005; Yan and Pollefeys, 2006; Elhamifar and Vidal, 2009; Peng et al., 2018; Meng et al., 2018). Arguably, a series of two-step algorithms, referring to Sparse Subspace Clustering (SSC) and its variants (Elhamifar and Vidal, 2009; Liu et al., 2012; Dyer et al., 2013; Heckel and Bölcskei, 2015; Chen et al., 2017), are the most popular and efficient methods for solving SC, which first construct a random graph (or an adjacent matrix equivalently), named as Union of Subspaces-based Random Geometry Graph (UoS-RGG), depending on the relative position among data points, and then apply the spectral method (Ng et al., 2002; Von Luxburg, 2007) to obtain the clustering result.

In spite of all these algorithms that practically work well for many applications, theoretical guarantees are lacked for the accuracy of clustering of any SC algorithm. We note that although novel and often efficient subspace clustering techniques emerge all the time, establishing rigorous theory for such techniques is quite difficult and does not exist as of now. The fundamental difficulty in the analysis of SC algorithms may be the change of view required in treating UoS-RGG (or general Random Geometry Graph, RGG), which has non-independent edges, in contrast with the traditional approach to analyzing clustering algorithms via Stochastic Block Model (SBM) which assumes independent edges. Section 1.2 offers a detailed discussion of this difficulty, as well as a survey of the existing attempts in theoretical aspects. We therefore propose the critical question that this paper aims to explore:

- Why does SC work, or more precisely, why does spectral method work for RGG or UoS-RGG?

This paper focuses on the analysis on the spectral method for UoS-RGG. We consider a naive and prototypical SC algorithm (Algorithm 1) here, and prove this algorithm, though oversimplified, can still deliver an almost correct clustering result even when the subspaces are quite close to each other and when the number of samples is far less than the subspace dimension (see Theorem 1). To the best of our knowledge, this is the first ever theory established to analyze the clustering error of SC algorithm. It not only constitutes the first theoretical guarantee for accuracy of subspace clustering, but also provides the interesting insight that the widely-conjectured oversampling requirement for subspace clustering is redundant, and that subspace clustering is quite robust in existence of closely aligned subspaces. We also verify our results by numerical experiments in Section 4. Although our theoretical results is proved only for the simplified algorithm we choose, it should be quite convincing that more carefully-designed SC algorithms would give even better performance than what we guarantee here, and our proof could serve as a prototype to the analysis of

these algorithms.

1.2 Related Works and Challenges

We now briefly review the literature on the adjacent matrix and spectral method and discuss their shortcomings. Since this paper mainly deals with theory, we shall focus on theoretical aspects of existing results.

1.2.1 Analysis of Random Graphs for UoS

To analyze the random graphs associated to UoS model in an abstract setting without referring to any specific algorithms, most researches focus on the Subspace Detection Property (SDP, Soltanolkotabi et al., 2012; Liu et al., 2012; Soltanolkotabi et al., 2014), a property which indicates that there are no edge connections between the data points in different subspaces, but are many connections between the data points in the same subspace. Under some technical conditions on the parameters of SC, the random graphs constructed by a variety of SC algorithms have been proved to enjoy SDP. Readers may consult Section 3 in Soltanolkotabi et al. (2014) for details.

There are, however, two main deficiencies of SDP which render SDP hard to use in further analysis. The first one is that SDP does not imply a correct clustering result. Actually, one can easily construct a counter-example where SDP holds but the clustering result is unsatisfying. The second one is that SDP requires too restrictive conditions on affinity between subspaces and sampling rate to hold. These conditions are provably unnecessary, as will be demonstrated in Section 3 of this paper.

1.2.2 Analysis of Spectral Method for Random Graphs

Compared with SDP, a more concrete approach to analyze SC algorithm is to investigate the performance of spectral method on random graphs associated to UoS model. To this end, analysis of spectral method for general random graphs (not necessarily associated to UoS model) is relevant. Note that the spectral method is explored deeply in the literature of community detection, which is an important problem in statistics, computer vision, and image processing (Abbe, 2017). Stochastic Block Model (SBM) is a widely used theoretical model in this field, which we briefly introduce as follows. For simplicity, we consider the two-block case, where the vertices of random graph are divided into two “blocks”, i.e. sets of vertices that ought to be closely-related, each of size of $N/2$. Then each edge of random graph is independently generated from the following distribution: for $p > q > 0$, vertices \mathbf{x}_i and \mathbf{x}_j are connected with probability p if $\mathbf{x}_i, \mathbf{x}_j$ belong to the same block, and with probability q if they belong to different blocks. Given an instance of this graph, we would

like to identify the two blocks. Recently, a series of theoretical works are devoted to analyze the performance of spectral method on this problem in different settings (Coja-Oghlan, 2010; Vu, 2014; Chin et al., 2015; Abbe et al., 2017), and extensions (Sankararaman and Baccelli, 2018).

As far as we know, all existing results make essential use of the independence of different edges, which is unfortunately not the case in SC algorithms. In fact, it is a generic and natural phenomenon in RGG that when $\mathbf{x}_i, \mathbf{x}_j$ and $\mathbf{x}_i, \mathbf{x}_k$ are connected, the probability that $\mathbf{x}_j, \mathbf{x}_k$ are connected will be higher, hence the independence assumption does not hold for RGG.

With this fundamental gap in mind, it is crucial to develop a theory for RGG to provide a rigorous theoretical guarantee for SC algorithms.

2 Preliminaries and Problem Formulation

The generative model for data points in UoS we adapt in this paper is the semi-random model introduced in Soltanolkotabi et al. (2012), which assumes that the subspaces are fixed with points distributed uniformly at random on each subspace. This is arguably the simplest model providing a good starting point for a theoretical investigation. We assume the data consists of two clusters, corresponding to two fixed subspaces¹ S_1, S_2 in \mathbb{R}^n , each with $N/2$ data points uniformly sampled from the unit spheres \mathcal{S}_1^{d-1} and \mathcal{S}_2^{d-1} respectively in S_1 and S_2 . Here d is the subspace dimension and n is the ambient dimension. The goal of SC is to cluster the normalized data points $\{\mathbf{x}_i\}_{1 \leq i \leq N}$.

Given the general description of SC, we turn our attention to a simple prototypical SC algorithm detailed in Algorithm 1, which we call Thresholding Inner-Product Subspace Clustering (TIP-SC). Considering that the angle between the data points in the same subspaces would be smaller statistically, we construct for some threshold $\tau \in (0, 1)$ the random graph by computing its adjacent matrix \mathbf{A} , where $A_{ij} = 1$ if $i \neq j, |\langle \mathbf{x}_i, \mathbf{x}_j \rangle| \geq \tau$, and $A_{ij} = 0$ otherwise. The TIP-SC algorithm concludes with applying the spectral clustering method on \mathbf{A} .

The main task of this paper is to prove this simple algorithm can achieve a high clustering accuracy under fairly general condition, which will be done in the next section.

Notations. Let $\mathbf{U}_1, \mathbf{U}_2$ denote the orthonormal bases for the subspaces S_1, S_2 , respectively, and $\lambda_1 \geq \dots \geq \lambda_d \geq 0$ denote the singular values of $\mathbf{U}_1^\top \mathbf{U}_2$. We also use S and S' to denote the subspaces to which \mathbf{x}_i does and doesn't belong, respectively. Then

¹It should be noticed that the number of subspaces is by no means crucial to the analysis. The results in this paper can be generalized to more subspaces easily.

Algorithm 1 Thresholding inner-product subspace clustering (TIP-SC)

Input: Normalized data set $\{\mathbf{x}_i\}_{1 \leq i \leq N}$, threshold τ .

- 1: **Construct Adjacent Matrix \mathbf{A} :**
 - 2: $A_{ij} = 1$ if $i \neq j, |\langle \mathbf{x}_i, \mathbf{x}_j \rangle| \geq \tau$, or $A_{ij} = 0$ otherwise.
 - 3: **Apply Spectral Method on \mathbf{A} :**
 - 4: Calculate \mathbf{W} , the eigenspace corresponding to the top two eigenvalues of \mathbf{A} .
 - 5: Use $\text{sgn}(\mathbf{w})$ as clustering result, where \mathbf{w} is the vector in \mathbf{W} perpendicular to the projection of all-ones vector in \mathbf{W} .
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$\mathbf{x}_i = \mathbf{U}\bar{\mathbf{a}}_i$ where \mathbf{U} denotes the orthonormal bases for S , $\mathbf{a}_i \stackrel{\text{ind.}}{\sim} \mathcal{N}(\mathbf{0}, \frac{1}{d}\mathbf{I}_d) \in \mathbb{R}^d$, and $\bar{\mathbf{a}}_i = \mathbf{a}_i/\|\mathbf{a}_i\|$ denotes its normalization. We use p, q to represent the probability that $A_{ij} = 1$ for $j \neq i, \mathbf{x}_j \in S$ and $\mathbf{x}_j \in S'$, respectively. Conditioned on \mathbf{x}_i , let p_i denote the probability of $A_{ij} = 1$ for $j \neq i, \mathbf{x}_j \in S$, and q_i denote the probability of $A_{ij} = 1$ for $j, \mathbf{x}_j \in S'$. Denote

$$\begin{aligned} \text{aff} &:= \sqrt{\frac{\sum_i \lambda_i^2}{d}}, \\ \kappa &:= 1 - \text{aff}^2, \\ \rho &:= \frac{N}{2d}. \end{aligned}$$

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^N$ with $u_i = \frac{1}{\sqrt{N}}$, and $v_i = \frac{1}{\sqrt{N}}$, if $\mathbf{x}_i \in S_1$, and $v_i = -\frac{1}{\sqrt{N}}$, if $\mathbf{x}_i \in S_2$, then \mathbf{v} is the ground truth. \mathbf{W} denotes the eigenspace corresponding to the top two eigenvalues of \mathbf{A} , and \mathbf{w} denotes the vector in \mathbf{W} , which is perpendicular to the projection of \mathbf{u} in \mathbf{W} .

3 Error Rate of TIP-SC Algorithm

This section presents our main theoretical results concerning the performance of TIP-SC. By the perturbation analysis of \mathbf{A} from $\mathbb{E}\mathbf{A}$, the success of spectral method for SBM has been proved in various statistical assumptions. However, such analysis is insufficient to establish our result, since for UoS-RGG, the independence condition doesn't hold, which is the crux leading to the failure of the existing methods for analyzing spectral method on random graph. As a substitute, we discover the conditional independence property for \mathbf{A} , based on which we prove that the clustering result of TIP-SC is almost correct under some mild condition on affinity and sampling rate, which is explained in the following theorem.

Theorem 1. Choosing $\tau = O\left(\frac{1}{\sqrt{d}}\right)$ such that $p = O(1)$, there exists some numerical constant $c > 0$, such that whenever $\kappa > c\sqrt{\frac{\log N}{d}}$, the clustering error rate of TIP-SC is less than $O\left(\frac{(1+1/\rho)\log N}{\kappa^2 d}\right)$ with probability at least $1 - e^{-\Omega(\log N)}$.

Parameter selection is often critical for the success of algorithms. The above result suggests that a dense graph ($p = O(1)$) is usually a good choice, which is quite different with SDP.

In this regime, the above result indicates that the algorithm works correctly in fairly broad conditions compared with existing analysis for SC. A fascinating insight revealed by the above theorem is that even when the number of samples $N \ll d$, we can succeed to cluster the data set, which demonstrates the commonly accepted opinion that $\rho > 1$ is necessary for SC is partially inaccurate.

To clarify the condition on κ , namely on affinity, assume these two subspaces overlap in a smaller subspace of dimension s , but are orthogonal to each other in the remaining directions. In this case, the affinity between the two subspaces is equal to $\sqrt{s/d}$. Our assumption on κ indicates that subspaces can have intersections of almost all dimensions, i.e., $s = (1 - o(1))d$. In contrast, previous works (Soltanolkotabi et al., 2012, 2014) imposes that the overlapping dimension should obey $s = o(1)d$, so that the subspaces are practically orthogonal to each other.

In the noisy case, we assume each data point is of the form

$$\mathbf{y} = \mathbf{x} + \mathbf{z}, \tag{1}$$

where \mathbf{x} denotes the clean data used in the above theorem, and $\mathbf{z} \sim \mathcal{N}(0, \frac{\sigma^2}{n} \mathbf{I})$ is an independent stochastic noise term. We have the following robustness guarantee for TIP-SC.

Theorem 2. Choosing $\tau = O\left(\frac{1}{\sqrt{d}}\right)$ such that $p = O(1)$, there exists some numerical constant $c, \sigma^* > 0$, such that whenever $\kappa > c\sqrt{\frac{\log N}{d}}$ and $\sigma < \sigma^*$, the clustering error rate of TIP-SC is less than $O\left(\frac{(1+\sigma^2 d/n)^2(1+1/\rho)\log N}{\kappa^2 d}\right)$ with probability at least $1 - e^{-\Omega(\log N)}$.

The proof is similar to that of Theorem 1, and both are deferred to Section 5.

4 Numerical Experiments

In this section, we perform numerical experiments validating our main results. We evaluate the algorithm and theoretical results based on the clustering accuracy. The impacts of κ, ρ, p, q on the clustering accuracy are demonstrated. Besides, we also show the efficiency of TIP-SC in the presence of noise.

According to the definition of semi-random model, to save computation and for simplicity, the data are generated by the following steps.

- 1) Given $d \ll n$ and $\text{aff} = \sqrt{s/d}$, define $\mathbf{e}_i \in \mathbb{R}^n$, whose entries are zero but the i -th entry is one. Let $\mathbf{U}_1 = [\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_d]$ be the orthonormal basis for subspace for S_1 ,

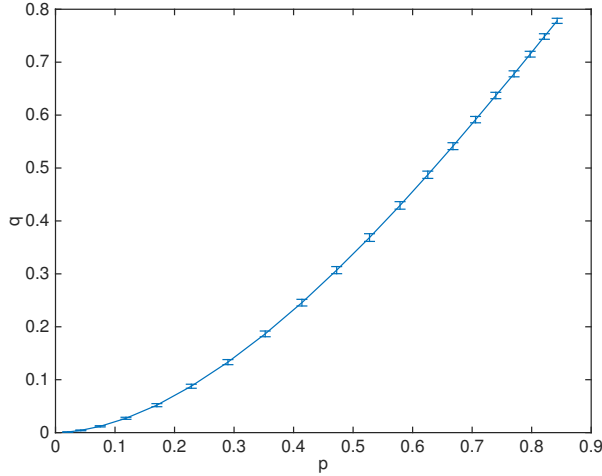


Figure 1: The relation between p and q , when $d = 100, n = 5000, \kappa = 1 - \sqrt{1/2}$ ($s = d/2$), $\rho = 1$.

and $\mathbf{U}_2 = [\mathbf{e}_{d-s+1}, \mathbf{e}_{d-s+2}, \dots, \mathbf{e}_{2d-s}]$ be the orthonormal basis for subspace for S_2 , such that the affinity between S_1 and S_2 is $\sqrt{s/d}$.

- 2) Given $N = \rho d$, generate N vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_N \in \mathbb{R}^d$ independently from $\mathcal{N}(0, \frac{1}{d}\mathbf{I})$. Let $\mathbf{x}_i = \mathbf{U}_1 \frac{\mathbf{a}_i}{\|\mathbf{a}_i\|}$ for $1 \leq i \leq N/2$ and $\mathbf{x}_i = \mathbf{U}_2 \frac{\mathbf{a}_i}{\|\mathbf{a}_i\|}$ for $N/2 + 1 \leq i \leq N$.
- 3) In the presence of noise, given $\sigma > 0$, generate N random noise terms $\mathbf{z}_1, \mathbf{z}_2, \dots, \mathbf{z}_N \in \mathbb{R}^n$ independently from $\mathcal{N}(0, \frac{\sigma^2}{n}\mathbf{I})$. Let the normalized data of $\mathbf{x}_i + \mathbf{z}_i$ be the input of Algorithm 1.

Since there are too many factors we need to consider, we always observe the relation between two concerned quantities, while keep others being some predefined typical values, i.e., $d^* = 100, n^* = 5000, \kappa^* = 1 - \sqrt{1/2}$ ($s^* = d/2$), $\rho^* = 1$, and τ is chosen to be τ^* such that the connection rate $\frac{p+q}{2} = 0.2$. We conduct the experiments in noiseless situations, except the last one which tests the robustness of Algorithm 1. Moreover, the curves are plotted by 100 trials in each experiment, while the mean and the standard deviation are represented by line and error bar, respectively. We can find that the randomness is eliminated in all experiments when the error rate is small.

It is obvious that p will decrease simultaneously if q decreases by increasing τ , which is also demonstrated in Figure 1. Combining the result of the second experiment (c.f. Figure 2), we can find that it is better to make p, q both large than to choose $q = 0$, although $q = 0$ is suggested by SDP, which is consistent with our result, while shows that SDP is somewhat inadequate for SC.

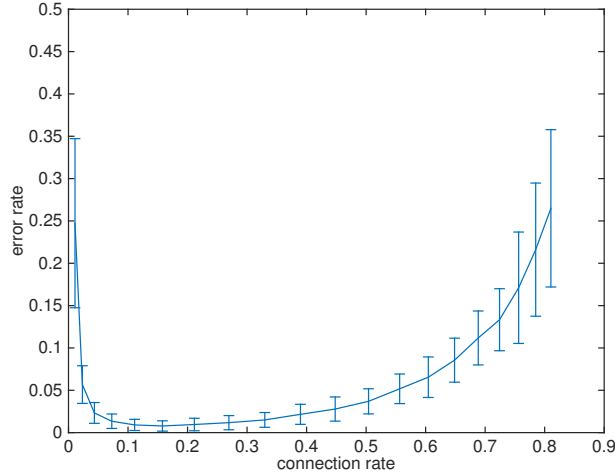


Figure 2: This figure demonstrates the clustering error rate versus the connection rate ($\frac{p+q}{2}$) in a general interval, when $d = 100$, $n = 5000$, $\kappa = 1 - \sqrt{1/2}(s = d/2)$, $\rho = 1$.

In the third and fourth experiments, we inspect the impacts of affinity and sampling rate on the performance of TIP-SC. From Figure 3 and Figure 4, the claim that SC works well in fairly broad conditions is verified. In addition, according to (1), we have

$$\text{SNR} = 10 \log \frac{1}{\sigma^2},$$

then the last experiment (c.f. Figure 5) shows that the algorithm is robust even though SNR is low.

5 Proof of Main Results

5.1 Proof of Theorem 1

Recall the definition of $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{W}$ in Section 2, and notice that analyzing the error rate, denoted by γ , is equivalent to studying the difference between \mathbf{w} and \mathbf{v} . Without loss of generality we may assume that $\langle \mathbf{w}, \mathbf{v} \rangle > 0$, thus the error rate is exactly

$$\gamma = \frac{1}{4} \left\| \frac{1}{\sqrt{N}} \text{sgn}(\mathbf{w}) - \mathbf{v} \right\|_2^2.$$

To estimate γ , it suffices to bound the distance between \mathbf{u}, \mathbf{v} and \mathbf{W} .

By simple geometric consideration, we have

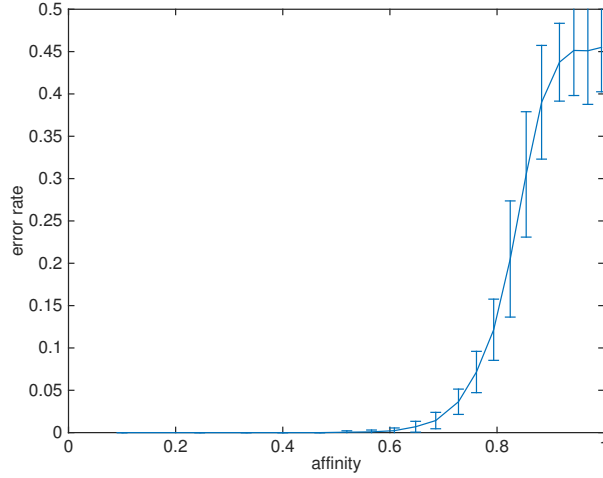


Figure 3: This figure demonstrates the clustering error rate versus the affinity in a general interval, when $d = 100$, $n = 5000$, $\rho = 1$, $\frac{p+q}{2} = 0.2$.

$$\begin{aligned}
\left\| \frac{1}{\sqrt{N}} \text{sgn}(\mathbf{w}) - \mathbf{v} \right\|_2 &\leq 2 \|\mathbf{P}_{\mathbf{w}} \mathbf{v} - \mathbf{v}\|_2 \\
&\leq 2(\|\mathbf{P}_{\mathbf{W}} \mathbf{v} - \mathbf{v}\|_2 + \|\mathbf{P}_{\mathbf{w}} \mathbf{v} - \mathbf{P}_{\mathbf{W}} \mathbf{v}\|_2) \\
&= 2(\|\mathbf{P}_{\mathbf{W}} \mathbf{v} - \mathbf{v}\|_2 + |\langle \overline{\mathbf{P}_{\mathbf{W}} \mathbf{u}}, \mathbf{v} \rangle|) \\
&\leq 2(\|\mathbf{P}_{\mathbf{W}} \mathbf{v} - \mathbf{v}\|_2 + \|\overline{\mathbf{P}_{\mathbf{W}} \mathbf{u}} - \mathbf{u}\|_2) \\
&\leq 2\|\mathbf{v} - \mathbf{P}_{\mathbf{W}} \mathbf{v}\|_2 + 4\|\mathbf{u} - \mathbf{P}_{\mathbf{W}} \mathbf{u}\|_2,
\end{aligned}$$

where $\overline{\mathbf{P}_{\mathbf{W}} \mathbf{u}}$ denote the normalization of $\mathbf{P}_{\mathbf{W}} \mathbf{u}$. Moreover, for any λ, \mathbf{x} , we have

$$\|\mathbf{A} \mathbf{x} - \lambda \mathbf{x}\|_2 \geq (\lambda - \lambda_3(\mathbf{A})) \|\mathbf{x} - \mathbf{P}_{\mathbf{W}} \mathbf{x}\|_2,$$

where $\lambda_3(\mathbf{A})$ denotes the third largest eigenvalue of \mathbf{A} .

Summing up, for $\lambda_1, \lambda_2 > \lambda_3(\mathbf{A})$,

$$\gamma = \frac{1}{4} \left\| \frac{1}{\sqrt{N}} \text{sgn}(\mathbf{w}) - \mathbf{v} \right\|_2^2 \lesssim \frac{\|\mathbf{A} \mathbf{u} - \lambda_1 \mathbf{u}\|_2^2}{(\lambda_1 - \lambda_3(\mathbf{A}))^2} + \frac{\|\mathbf{A} \mathbf{v} - \lambda_2 \mathbf{v}\|_2^2}{(\lambda_2 - \lambda_3(\mathbf{A}))^2},$$

Considering that $\mathbb{E} \langle \mathbf{A} \mathbf{u}, \mathbf{u} \rangle = p(N/2 - 1) + qN/2$, we expect $\lambda_1 = p(N/2 - 1) + qN/2$ is a good choice. Similarly, choose $\lambda_2 = p(N/2 - 1) - qN/2$.

From above discussion, to estimate γ we need to:

- Prove $\|\mathbf{A} \mathbf{u} - \lambda_1 \mathbf{u}\|_2$ and $\|\mathbf{A} \mathbf{v} - \lambda_2 \mathbf{v}\|_2$ are sufficiently small (see Lemma 3 and Lemma 4).

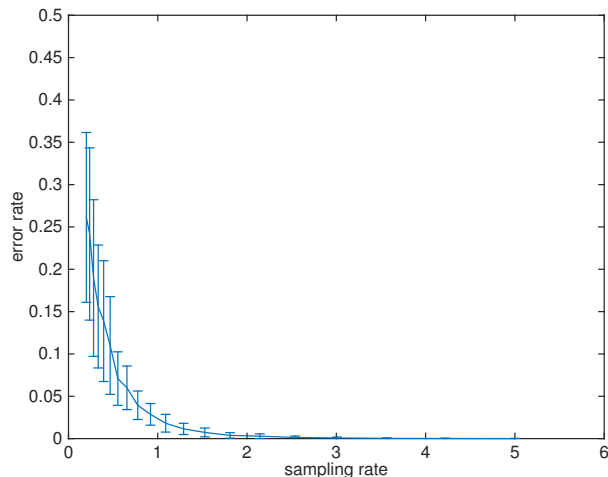


Figure 4: This figure demonstrates the clustering error rate versus the sampling rate ρ in a general interval, when $d = 100, n = 5000, \kappa = 1 - \sqrt{1/2}$ ($s = d/2$), $\frac{p+q}{2} = 0.2$.

- Prove $\lambda_1 - \lambda_3(\mathbf{A})$ and $\lambda_2 - \lambda_3(\mathbf{A})$ are sufficiently large, which is equivalent to showing $p - q$ is large enough (see Lemma 3) and $\lambda_3(\mathbf{A})$ is small enough (see Lemma 5).

Before proceeding, we analyze the adjacent matrix \mathbf{A} based on the conditional independence property, and provide probability estimations used in the proof of Theorem 1. Specifically, this refers to if conditioned on $\mathbf{x}_i, i \in \mathcal{S}$ for some subset \mathcal{S} of $[N]$, A_{ij} , for $j \in \mathcal{S}^c$, are functions of \mathbf{x}_j , respectively, and then are independent from each other.

Moreover, recalling the definition of $\mathbf{x}_i, \mathbf{a}_i$, on the collection of events $\mathcal{E}(t)$ given by the intersection of

$$\begin{aligned} & \{\forall i, \|\mathbf{a}_i\| - 1 < t\} \\ & \left\{ \forall i, \left| \sum_k \lambda_k^2 a_{ik}^2 - \frac{\sum_k \lambda_k^2}{d} \right| < t \right\} \\ & \{\forall i \neq j, |\langle \mathbf{x}_i, \mathbf{x}_j \rangle| < t\}, \end{aligned}$$

if conditioned on $\mathbf{x}_i, i \in \mathcal{S}$, A_{ij} , for $j \in \mathcal{S}^c$ are *nearly identically distributed*, and for some $j \in \mathcal{S}^c$, A_{ij} , for $i \in \mathcal{S}$ are *nearly independent* from each other, which will be explained and employed many times in the following analysis. According to Lemma 7 and Lemma 8, there exist some constants $c_1, c_2 > 0$, such that

$$\mathbb{P} \left(\mathcal{E} \left(c_1 \sqrt{\frac{\log N}{d}} \right) \right) > 1 - e^{-c_2 \log N}.$$

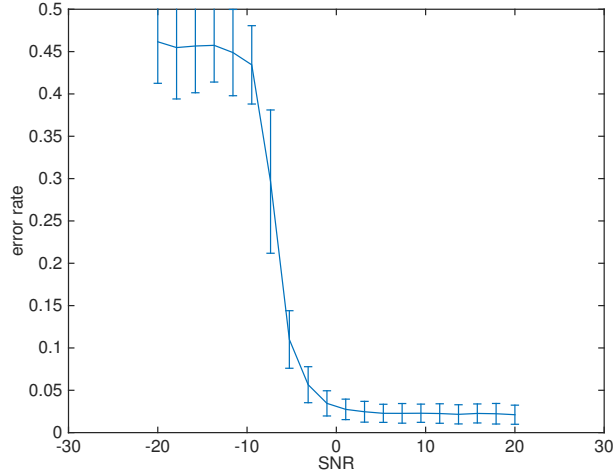


Figure 5: This figure demonstrates the clustering error rate versus the SNR in a general interval, when $d = 100, n = 5000, \kappa = 1 - \sqrt{1/2}$ ($s = d/2$), $\rho = 1, \frac{p+q}{2} = 0.2$.

For simplicity, use \mathcal{E} to denote $\mathcal{E}\left(c_1\sqrt{\frac{\log N}{d}}\right)$. In this work, we will always analyze the spectral method on the canonical event set \mathcal{E} .

Let

$$\Phi(t) := \int_{|x|>t} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx,$$

then

Lemma 1. All p_i are equal, and there exist constants $c_1, c_2 > 0$, such that

$$p = p_i > \Phi(\tau_d(1+t)) - e^{-c_2 \log N},$$

where $\tau_d := \sqrt{d}\tau$ and $t = c_1\sqrt{\frac{\log N}{d}}$.

Proof. Conditioned on \mathbf{x}_i , for $\mathbf{x}_j \in S$

$$p_i := \mathbb{P}\left(|\langle \mathbf{x}_i, \mathbf{x}_j \rangle| \geq \tau \mid \mathbf{x}_i\right) = \mathbb{P}\left(|\langle \bar{\mathbf{a}}_i, \bar{\mathbf{a}}_j \rangle| \geq \tau \mid \mathbf{a}_i\right),$$

where $\mathbf{a}_i, \mathbf{a}_j \stackrel{\text{ind.}}{\sim} \mathcal{N}(\mathbf{0}, \frac{1}{d}\mathbf{I}_d) \in \mathbb{R}^d$, and $\bar{\mathbf{a}}_i, \bar{\mathbf{a}}_j$ denote the normalization with $\mathbf{x}_i = \mathbf{U}\bar{\mathbf{a}}_i, \mathbf{x}_j = \mathbf{U}\bar{\mathbf{a}}_j$. According to the independence between $\mathbf{a}_i, \mathbf{a}_j$ and the rotational invariance property

of Gaussian random vectors, it is obviously that all p_i are equal. Moreover, we have

$$\begin{aligned}
& \mathbb{P} \left(|\langle \mathbf{x}_i, \mathbf{x}_j \rangle| \geq \tau \middle| \mathbf{x}_i \right) \\
&= 2\mathbb{P} \left(\langle \bar{\mathbf{a}}_i, \mathbf{a}_j \rangle \geq \tau \|\mathbf{a}_j\| \middle| \mathbf{a}_i \right) \\
&\geq 2\mathbb{P} \left(\langle \bar{\mathbf{a}}_i, \mathbf{a}_j \rangle \geq \tau(1+t) \middle| \mathbf{a}_i \right) - 2\mathbb{P} (\|\mathbf{a}_j\| > 1+t) \\
&> \Phi \left(\sqrt{d}\tau(1+t) \right) - e^{-c_2 \log N},
\end{aligned}$$

since $\langle \bar{\mathbf{a}}_i, \mathbf{a}_j \rangle \sim \mathcal{N}(0, \frac{1}{d})$ is a Gaussian random variable independent with \mathbf{a}_i , and

$$\mathbb{P} (\|\mathbf{a}_j\| > 1+t) < e^{-c_2 \log N}$$

according to Lemma 7. □

Lemma 2. There exist constants $c_1, c_2 > 0$, such that for $t = c_1 \sqrt{\frac{\log N}{d}}$, on $\mathcal{E}(t)$, we have

$$\Phi \left(\frac{\tau_d(1+t)}{\text{aff}^2 - t} \right) - e^{-c_2 \log N} < q_i < \Phi \left(\frac{\tau_d(1-t)}{\text{aff}^2 + t} \right) + e^{-c_2 \log N},$$

where $\tau_d := \sqrt{d}\tau$.

Proof. According to Remark 5 in Li and Gu (2017), we can choose $\mathbf{U}_1, \mathbf{U}_2$ such that

$$\mathbf{U}_2^\top \mathbf{U}_1 = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix}.$$

Without loss of generality, assume that $\mathbf{x}_i \in S_1, \mathbf{x}_j \in S_2$, then

$$\mathbb{P} \left(|\langle \mathbf{x}_i, \mathbf{x}_j \rangle| \geq \tau \middle| \mathbf{x}_i \right) = \mathbb{P} \left(|\langle \mathbf{U}_1 \bar{\mathbf{a}}_i, \mathbf{U}_2 \bar{\mathbf{a}}_j \rangle| \geq \tau \middle| \mathbf{a}_i \right),$$

where $\mathbf{a}_i, \mathbf{a}_j \stackrel{\text{ind.}}{\sim} \mathcal{N}(\mathbf{0}, \frac{1}{d} \mathbf{I}_d) \in \mathbb{R}^d$, and $\bar{\mathbf{a}}_i, \bar{\mathbf{a}}_j$ denote the normalization with $\mathbf{x}_i = \mathbf{U}_1 \bar{\mathbf{a}}_i, \mathbf{x}_j = \mathbf{U}_2 \bar{\mathbf{a}}_j$. In addition, the definition of $\mathcal{E}(t)$ gives,

$$\left| \|\mathbf{U}_2^\top \mathbf{U}_1 \bar{\mathbf{a}}_i\|^2 - \text{aff}^2 \right| < t,$$

then according to Lemma 7

$$\begin{aligned}
q_i &= \mathbb{P} \left(|\langle \mathbf{x}_i, \mathbf{x}_j \rangle| \geq \tau \middle| \mathbf{x}_i \right) \\
&= \mathbb{P} \left(|\langle \mathbf{U}_2^\top \mathbf{U}_1 \bar{\mathbf{a}}_i, \bar{\mathbf{a}}_j \rangle| \geq \tau \middle| \mathbf{a}_i \right) \\
&\geq 2\mathbb{P} \left(\langle \mathbf{U}_2^\top \mathbf{U}_1 \bar{\mathbf{a}}_i, \mathbf{a}_j \rangle \geq \tau(1+t) \middle| \mathbf{a}_i \right) - 2\mathbb{P} (\|\mathbf{a}_j\| > 1+t) \\
&> \Phi \left(\frac{\sqrt{d}\tau(1+t)}{\text{aff}^2 - t} \right) - e^{-c_2 \log N},
\end{aligned}$$

and similarly,

$$q_i < \Phi \left(\frac{\sqrt{d}\tau(1-t)}{\text{aff}^2 + t} \right) + e^{-c_2 \log N}.$$

□

Specifically, according to the above two lemmas about p_i, q_i , we can easily get the following lemma.

Lemma 3. Choose $\tau_d = O(1)$, then $p = \Omega(1)$. Moreover, on \mathcal{E} , there exists some constant $c > 0$, such that if $\kappa = 1 - \text{aff}^2 > c\sqrt{\frac{\log N}{d}}$,

$$p - q \gtrsim \kappa,$$

and

$$\frac{1}{N} \sum_i (q_i - q)^2 \lesssim \frac{\log N}{d}.$$

Having finished the calculation about the probability of each entry, we now turn to the overall properties of \mathbf{A} .

Lemma 4. Conditioned on \mathbf{x}_i , for any $t > 0$

$$\mathbb{P} \left(\left| \frac{1}{N/2-1} \sum_{j:\mathbf{x}_j \in S} A_{ij} - p \right| > t \right) < e^{-\frac{t^2(N/2-1)}{p+\frac{1}{3}t}},$$

and

$$\mathbb{P} \left(\left| \frac{1}{N/2} \sum_{j:\mathbf{x}_j \in S'} A_{ij} - q_i \right| > t \right) < e^{-\frac{t^2 N/2}{q_i + \frac{1}{3}t}}.$$

Proof. Given \mathbf{x}_i , it can be easily checked that the angles between \mathbf{x}_j and \mathbf{x}_i are independent with each other, then A_{ij} are conditionally independent Bernoulli random variables. Hence, according to Lemma 9, the results is obvious. □

In the next lemma, we will analyze the eigenvalue of \mathbf{A} .

Lemma 5. For $t = c_1 \sqrt{\frac{\log N}{d}}$, on $\mathcal{E}(t)$, with probability at least $1 - e^{-c_2 \log N}$,

$$\lambda_3(\mathbf{A}) < c\sqrt{Np \log N + N^2 p^2 t},$$

where $\lambda_3(\mathbf{A})$ denotes the third largest eigenvalue of \mathbf{A} , and $c, c_1, c_2 > 0$ are some constants.

Proof. We transfer the estimation of $\lambda_3(\mathbf{A})$ to bounding $\lambda_{\max}(\mathbf{E})$ using Lemma 10, i.e.,

$$\begin{aligned}
\lambda_3(\mathbf{A}) &= \min_{S_{N-2}} \max_{\mathbf{x} \in S_{N-2}} \mathbf{x}^\top \mathbf{A} \mathbf{x} \\
&\leq \max_{\mathbf{x} \in \text{span}(\mathbf{u}, \mathbf{v})^\perp} \mathbf{x}^\top \mathbf{A} \mathbf{x} \\
&= \max_{\mathbf{x} \in \text{span}(\mathbf{u}, \mathbf{v})^\perp} \mathbf{x}^\top \mathbf{E} \mathbf{x} \\
&\leq \max_{\mathbf{x}: \|\mathbf{x}\|=1} \mathbf{x}^\top \mathbf{E} \mathbf{x} \\
&= \lambda_{\max}(\mathbf{E}),
\end{aligned}$$

where \mathbf{u}, \mathbf{v} are defined in Section 2, and

$$\mathbf{E} = \mathbf{A} - (p+q)N/2\mathbf{u}\mathbf{u}^\top - (p-q)N/2\mathbf{v}\mathbf{v}^\top,$$

then $E_{ij} = -p$, if $i = j$, $E_{ij} = A_{ij} - p$, if $\mathbf{x}_j \in S$ and $E_{ij} = A_{ij} - q$, if $\mathbf{x}_j \in S'$.

The analysis of $\lambda_{\max}(\mathbf{E})$ is based on the decoupling technique. According to Lemma 11, let \mathcal{S} be a random subset of $[N]$ with average size $N/2$, then

$$\begin{aligned}
\lambda_{\max}(\mathbf{E}) &= \sup_{\mathbf{x}: \|\mathbf{x}\|=1} \mathbf{x}^\top \mathbf{E} \mathbf{x} \\
&= \sup_{\mathbf{x}: \|\mathbf{x}\|=1} \left(\sum_i x_i^2 E_{ii} + \sum_{i \neq j} x_i x_j E_{ij} \right) \\
&= -p + \sup_{\mathbf{x}: \|\mathbf{x}\|=1} 4\mathbb{E}_{\mathcal{S}} \sum_{i \in \mathcal{S}, j \in \mathcal{S}^c} x_i x_j E_{ij} \\
&\leq -p + 4\mathbb{E}_{\mathcal{S}} \sup_{\mathbf{x}: \|\mathbf{x}\|=1} \sum_{i \in \mathcal{S}, j \in \mathcal{S}^c} x_i x_j E_{ij} \\
&\leq -p + 4\mathbb{E}_{\mathcal{S}} \|\mathbf{E}_{\mathcal{S}, \mathcal{S}^c}\|_{\text{op}},
\end{aligned}$$

where $\mathbf{E}_{\mathcal{S}, \mathcal{S}^c}$ denotes the sub-matrix of \mathbf{E} including the rows from \mathcal{S} and columns from \mathcal{S}^c , and $\|\cdot\|_{\text{op}}$ denotes the operator norm.

To analyze $\|\mathbf{E}_{\mathcal{S}, \mathcal{S}^c}\|_{\text{op}}$, we first condition on \mathcal{S} and $\mathbf{x}_j, j \in \mathcal{S}^c$, and for $i \in \mathcal{S}$, let $\Gamma_i := \mathbf{E}_{i, \mathcal{S}^c}, \mathbf{R}_i := \mathbb{E}\Gamma_i^\top \Gamma_i$, and $L := \max_i \|\Gamma_i\|^2$, then Γ_i are independent with each other. On \mathcal{E} ,

$$\begin{aligned}
L &= \max_i \|\Gamma_i\|^2 \\
&= \max_i \sum_{j \in \mathcal{S}^c} E_{ij}^2 \\
&\lesssim Np.
\end{aligned}$$

Moreover, for the diagonal entries of \mathbf{R}_i ,

$$\begin{aligned}
\mathbf{x}_j \in S : \mathbb{E}\Gamma_{ij}\Gamma_{ij} &= \mathbb{E}(A_{ij} - p)^2 = p(1-p) \leq p, \\
\mathbf{x}_j \in S' : \mathbb{E}\Gamma_{ij}\Gamma_{ij} &= \mathbb{E}(A_{ij} - q)^2 = q(1-q) + (q_j - q)^2 \leq p.
\end{aligned}$$

On the other hand, for the off-diagonal entries of \mathbf{R}_i , if $\mathbf{x}_j, \mathbf{x}_k \in S$,

$$|\mathbb{E}\Gamma_{ij}\Gamma_{ik}| = |\mathbb{E}(A_{ij} - p)(A_{ik} - p)| \lesssim p^2t,$$

since $\langle \mathbf{x}_j, \mathbf{x}_k \rangle \leq t$. With similar analysis on the cases $\mathbf{x}_j \in S', \mathbf{x}_k \in S$ and $\mathbf{x}_j, \mathbf{x}_k \in S'$, we have the off-diagonal entries of \mathbf{R}_i are less than p^2t . Hence,

$$\lambda_{\max}(\mathbf{R}_i) \lesssim p + Np^2t =: \lambda.$$

and Lemma 12 gives, for $0 < \theta < 3/L$,

$$\begin{aligned} \log \mathbb{E} \exp \left(\theta \sum_i \left(\Gamma_i^\top \Gamma_i - \mathbf{R}_i \right) \right) &= \sum_i \log \mathbb{E} \exp \left(\theta \left(\Gamma_i^\top \Gamma_i - \mathbf{R}_i \right) \right) \\ &\asymp \sum_i \frac{\theta^2/2}{1 - \theta L/3} \mathbb{E} \left(\Gamma_i^\top \Gamma_i - \mathbf{R}_i \right)^2 \\ &\asymp \sum_i \frac{\theta^2/2}{1 - \theta L/3} L \mathbf{R}_i. \end{aligned}$$

Then

$$\begin{aligned} &\mathbb{P} \left(\mathbb{E}_S \left(\|\mathbf{E}_{S, S^c}\|_{\text{op}}^2 - \sum_i \lambda_{\max}(\mathbf{R}_i) \right) > t \right) \\ &< \inf_{\theta} e^{-\theta t} \mathbb{E} \exp \left(\mathbb{E}_S \theta \left(\|\mathbf{E}_{S, S^c}\|_{\text{op}}^2 - \sum_i \lambda_{\max}(\mathbf{R}_i) \right) \right) \\ &\leq \inf_{\theta} e^{-\theta t} \mathbb{E} \mathbb{E}_S \exp \left(\theta \left(\|\mathbf{E}_{S, S^c}\|_{\text{op}}^2 - \sum_i \lambda_{\max}(\mathbf{R}_i) \right) \right) \\ &\leq \inf_{\theta} e^{-\theta t} \mathbb{E} \text{Tr} \left(\exp \left(\theta \sum_i \left(\Gamma_i^\top \Gamma_i - \mathbf{R}_i \right) \right) \right) \\ &\leq \inf_{\theta} e^{-\theta t} \mathbb{E} \text{Tr} \left(\exp \left(\sum_i \frac{\theta^2/2}{1 - \theta L/3} L \mathbf{R}_i \right) \right) \\ &\leq \inf_{\theta} e^{-\theta t} N \exp \left(\sum_i \frac{\theta^2/2}{1 - \theta L/3} L \lambda_{\max}(\mathbf{R}_i) \right) \\ &\lesssim N \exp \left(\frac{-t^2/2}{NL\lambda + Lt/3} \right). \end{aligned}$$

Hence, with probability at least $1 - e^{-c_2 \log N}$,

$$\begin{aligned} \mathbb{E}_S \|\mathbf{E}_{S, S^c}\|_{\text{op}} &\leq \sqrt{\mathbb{E}_S \|\mathbf{E}_{S, S^c}\|_{\text{op}}^2} \\ &\leq c \sqrt{N\lambda + L \log N} \\ &= c \sqrt{Np \log N + N^2 p^2 t}. \end{aligned}$$

Summing up,

$$\lambda_3(\mathbf{A}) \leq \lambda_{\max}(\mathbf{E}) < c\sqrt{Np \log N + N^2 p^2 t}.$$

We conclude the proof. \square

Now, we have all the ingredients for the proof of Theorem 1.

Proof of Theorem 1. We begin with some inequalities for estimating the error. We have

$$\begin{aligned} & \|\mathbf{A}\mathbf{u} - (p(N/2 - 1) + qN/2)\mathbf{u}\|_2^2 \\ &= \frac{1}{N} \sum_i \left(\sum_{j:\mathbf{x}_j \in S} A_{ij} - p(N/2 - 1) + \sum_{j:\mathbf{x}_j \in S'} A_{ij} - qN/2 \right)^2 \\ &\leq \frac{3}{N} \sum_i \left(\sum_{j:\mathbf{x}_j \in S} A_{ij} - p(N/2 - 1) \right)^2 \\ &\quad + \frac{3}{N} \sum_i \left(\sum_{j:\mathbf{x}_j \in S'} A_{ij} - q_i N/2 \right)^2 + \frac{3}{N} \sum_i (q_i N/2 - qN/2)^2. \end{aligned}$$

According to Lemma 4, for all $1 \leq i \leq N$, we have, with probability at least $1 - \exp(-\Omega(\log N))$,

$$\left(\sum_{j:\mathbf{x}_j \in S} A_{ij} - p(N/2 - 1) \right)^2 \lesssim N \log N,$$

and

$$\left(\sum_{j:\mathbf{x}_j \in S'} A_{ij} - q_i N/2 \right)^2 \lesssim N \log N.$$

On the other hand, Lemma 3 gives, with probability at least $1 - \exp(-\Omega(\log N))$,

$$\frac{3}{N} \sum_i (q_i N/2 - qN/2)^2 \lesssim \rho N \log N.$$

Summing up, we have, with probability at least $1 - \exp(-\Omega(\log N))$,

$$\|\mathbf{A}\mathbf{u} - (p(N/2 - 1) + qN/2)\mathbf{u}\|_2^2 \lesssim (1 + \rho)N \log N.$$

Similarly, with probability at least $1 - \exp(-\Omega(\log N))$,

$$\|\mathbf{A}\mathbf{v} - (p(N/2 - 1) - qN/2)\mathbf{v}\|_2^2 \lesssim (1 + \rho)N \log N.$$

According to Lemma 5, for $t = O\left(\sqrt{\frac{\log N}{d}}\right)$, with probability at least $1 - \exp(-\Omega(\log N))$, the third largest eigenvalue of \mathbf{A} satisfies

$$\lambda_3(\mathbf{A}) \lesssim \sqrt{Np \log N + N^2 p^2 t} = O\left(N\sqrt[4]{\frac{\log N}{d}}\right).$$

With these estimations at hand, recall

$$\gamma \lesssim \frac{\|\mathbf{A}\mathbf{u} - (p(N/2 - 1) + qN/2)\mathbf{u}\|_2^2}{|p(N/2 - 1) + qN/2 - \lambda_3(\mathbf{A})|^2} + \frac{\|\mathbf{A}\mathbf{v} - (p(N/2 - 1) - qN/2)\mathbf{v}\|_2^2}{|p(N/2 - 1) - qN/2 - \lambda_3(\mathbf{A})|^2}.$$

Lemma 3 gives $p \pm q \gtrsim 1 - \text{aff}^2$, then we have

$$\gamma \lesssim \left(\frac{\sqrt{(1 + \rho)N \log N}}{N \left(1 - \text{aff}^2 - \sqrt[4]{\frac{\log N}{d}} \right)} \right)^2 \sim \frac{(1 + \rho) \log N}{\kappa^2 N}.$$

We conclude the proof. \square

5.2 Proof of Theorem 2

Robustness analysis can be completed by following the similar analysis method. We provide the differences in the analysis of noise, while omit the details.

Here, we only need to pay attention to the changes of Lemma 3, Lemma 4, and Lemma 5, when adding noise. Notice that the noise terms do not destroy the wonderful conditional independence property, then it's obvious that except the estimation for $p - q$, all other bounds still hold in a similar way. Through simple calculation, the contribution of noise has the form

$$p - q \gtrsim \frac{\kappa}{1 + \sigma^2 d/n}.$$

Taking this change into account, we can get the result easily.

6 Conclusion

This paper establish a theory to analyze spectral method for Random Geometry Graph constructed by data points from Union of Subspaces. Based on this theory, we demonstrate the efficiency of Subspace Clustering in fairly broad conditions. To the best of our knowledge, the clustering accuracy has not been shown in the prior literature. The insights and analysis techniques developed in this paper might also have implications for other Random Geometry Graph.

Moving forward, one issue is to understand UoS-RGG constructed by more complex strategy, such as SSC. Additionally, ideally one would desire an exact recovery by spectral method, which needs entrywise analysis. We leave these to future investigation.

A Auxiliary Lemmas

In this subsection, we introduce some well-known results about Gaussian, Bernoulli random variables, and matrices (Vershynin, 2010), which shall be used to analyze the properties of

the adjacent matrix \mathbf{A} . We omit the proof for most of them.

Lemma 6 (Concentration in Gauss space (Ledoux, 2001)). Let f be a real valued Lipschitz function on \mathbb{R}^n with Lipschitz constant K , i.e.,

$$|f(\mathbf{x}_1) - f(\mathbf{x}_2)| \leq K \|\mathbf{x}_1 - \mathbf{x}_2\|$$

for any $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$ (such functions are also called K -Lipschitz). Let $X \sim \mathcal{N}(\mathbf{0}, \mathbf{I}_n)$ be the standard Gaussian random vector in \mathbb{R}^n , then for every $t > 0$, one has

$$\mathbb{P}(f(\mathbf{x}) - \mathbb{E}f(\mathbf{x}) > t) < e^{-\frac{t^2}{2K^2}}.$$

Lemma 7. Assume $\mathbf{a} \sim \mathcal{N}(\mathbf{0}, \frac{1}{d}\mathbf{I}_d) \in \mathbb{R}^d$, then for any $t > 0$

$$\mathbb{P}\left(\left|\|\mathbf{a}\| - 1\right| > t + \frac{2}{\sqrt{d}}\right) < 2e^{-\frac{dt^2}{2}}. \quad (2)$$

Moreover, for $0 \leq \lambda_1, \dots, \lambda_d \leq 1$ and $t > 0$

$$\mathbb{P}\left(\left|\sum_i \lambda_i^2 a_i^2 - \frac{\sum_i \lambda_i^2}{d}\right| > 2t\sqrt{\frac{\sum_i \lambda_i^2}{d}} + t^2 + \frac{4}{d}\right) < 2e^{-\frac{dt^2}{2}}. \quad (3)$$

Proof. Let

$$f(\mathbf{x}) = \sqrt{\sum_i \lambda_i^2 x_i^2},$$

then by calculation

$$\|\nabla f(\mathbf{x})\| = \sqrt{\frac{\sum_i \lambda_i^4 x_i^2}{\sum_i \lambda_i^2 x_i^2}} \leq 1.$$

Hence, $f(\mathbf{x})$ is 1-Lipschitz and according to Lemma 6, we have

$$\mathbb{P}\left(\sqrt{\sum_i \lambda_i^2 a_i^2} - \mathbb{E}\sqrt{\sum_i \lambda_i^2 a_i^2} > t\right) < e^{-\frac{dt^2}{2}}.$$

Take $f(x) = -\sqrt{\sum_i \lambda_i^2 x_i^2}$, then similarly

$$\mathbb{P}\left(\sqrt{\sum_i \lambda_i^2 a_i^2} - \mathbb{E}\sqrt{\sum_i \lambda_i^2 a_i^2} < -t\right) < e^{-\frac{dt^2}{2}}.$$

Moreover, $\left(\mathbb{E}\sqrt{\sum_i \lambda_i^2 a_i^2}\right)^2 \leq \mathbb{E} \sum_i \lambda_i^2 a_i^2 = \frac{\sum_i \lambda_i^2}{d}$ and

$$\begin{aligned}
& \left(\mathbb{E}\sqrt{\sum_i \lambda_i^2 a_i^2}\right)^2 \\
&= \mathbb{E} \sum_i \lambda_i^2 a_i^2 - \text{Var}\left(\sqrt{\sum_i \lambda_i^2 a_i^2}\right) \\
&= \frac{\sum_i \lambda_i^2}{d} - \int_t t^2 d\mathbb{P}\left(\left|\sqrt{\sum_i \lambda_i^2 a_i^2} - \mathbb{E}\sqrt{\sum_i \lambda_i^2 a_i^2}\right| < t\right) \\
&= \frac{\sum_i \lambda_i^2}{d} - \int_t 2t\mathbb{P}\left(\left|\sqrt{\sum_i \lambda_i^2 a_i^2} - \mathbb{E}\sqrt{\sum_i \lambda_i^2 a_i^2}\right| > t\right) dt \\
&\geq \frac{\sum_i \lambda_i^2 - 4}{d}.
\end{aligned}$$

Taking $\lambda_i = 1$, we prove (2). Taking square, we prove (3). \square

Here, we also use $\langle \mathbf{a}, \mathbf{b} \rangle$ to denote the angle between \mathbf{a} and \mathbf{b} .

Lemma 8 (Concentration of measure (Ledoux, 2001)). Assume $\mathbf{a}, \mathbf{b} \stackrel{\text{ind.}}{\sim} \mathcal{N}(\mathbf{0}, \frac{1}{d}\mathbf{I}_d) \in \mathbb{R}^d$, then for any $t > 0$

$$\mathbb{P}(|\cos\langle \mathbf{a}, \mathbf{b} \rangle| > t) < 2e^{-\frac{dt^2}{2}}.$$

Lemma 9. X_1, X_2, \dots, X_N are generated independently from $\text{Bern}(p)$, then for any $t > 0$

$$\mathbb{P}\left(\left|\frac{1}{N} \sum_i X_i - p\right| > t\right) < e^{-\frac{t^2 N}{p + \frac{1}{3}t}}.$$

Proof. According to Bernstein's Inequality, the conclusion is obvious. \square

Lemma 10. For any symmetric matrix $\mathbf{M} \in \mathbb{R}^{n \times n}$,

$$\lambda_{i+1} = \min_{S_{n-i}} \max_{\mathbf{x} \in S_{n-i}} \mathbf{x}^\top \mathbf{M} \mathbf{x},$$

where S_{n-i} denotes the subspace of \mathbb{R}^n of dimension $n - i$.

Proof. This is a basic property of eigenvalues. \square

We define a random subset \mathcal{S} of $[N]$ with average size αN as follows. For all $i \in [N]$, i belongs to \mathcal{S} with probability α independently from each other. Then we state an elementary decoupling lemma for double arrays here.

Lemma 11 (Decoupling (Helmers, 2000)). Consider a double array of real numbers $(a_{ij})_{i,j=1}^{2N}$ such that $a_{ii} = 0$ for all i . Then

$$\sum_{i,j \in [N]} a_{ij} = 4\mathbb{E} \sum_{i \in \mathcal{S}, j \in \mathcal{S}^c} a_{ij},$$

where \mathcal{S} is a random subset of $[N]$ with average size $N/2$.

Lemma 12 (Matrix Bernstein: Mgf and Cgf Bound, Lemma 6.6.2 (Tropp et al., 2015)). Suppose that \mathbf{X} is a random Hermitian matrix that satisfies

$$\mathbb{E}\mathbf{X} = 0, \lambda_{\max}(\mathbf{X}) \leq L,$$

then for $0 < \theta < 3/L$

$$\log \mathbb{E}e^{\theta\mathbf{X}} \preceq \frac{\theta^2/2}{1 - \theta L/3} \mathbb{E}\mathbf{X}^2.$$

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