Homework#2: Short course on sparse recovery and compressed sensing

1. **Answer:** Two lectures were given today. The first lecture introduces the uncertainty principle between the identity basis and the Fourier basis, which ensures that sparse combinations of these dictionary elements can only be written one way with a small number of terms. Generally, this uncertainty principle can be characterized by the concept of coherence, which greatly affects the performance of sparse representations under the overcomplete dictionary. The second lecture gives a high-level survey of compressive sensing and its applications. Compressive sensing is a new sampling theory that leverages compressibility of signals to replace front-end acquisition complexity with back-end computing. The key roles played are the new uncertainty principle and randomness. A bunch of applications are introduced to enhance the understanding of how compressive sensing works.

2. **Answer:** According to the discrete uncertainty principle, letting $|T|$ denote the size of the support of f and $|\Omega|$ denote the size of the support of $\hat{f}$, the following relationship holds

$$|T| \cdot |\Omega| \geq n,$$

where $n = 16$ in this problem setting.

For pair (a), $|T| = 4$ and $|\Omega| = 4$, which satisfies $|T| \cdot |\Omega| \geq n$.

For pair (b), $|T| = 1$ and $|\Omega| = 16$, which satisfies $|T| \cdot |\Omega| \geq n$

For pair (c), $|T| = 3$ and $|\Omega| = 3$, which does not satisfy $|T| \cdot |\Omega| \geq n$.

For pair (d), $|T| = 16$ and $|\Omega| = 16$, which satisfies $|T| \cdot |\Omega| \geq n$.

For pair (e), $|T| = 2$ and $|\Omega| = 8$, which satisfies $|T| \cdot |\Omega| \geq n$.

Therefore, (a), (b), (d), and (e) are possible time/frequency pairs.

3. **Answer:**

   (a) It can be calculated that

$$\begin{aligned}
\Phi_\Gamma^* \Phi_\Gamma &= \begin{bmatrix} \Phi_{\Gamma_1}^{1*} \\ \Phi_{\Gamma_2}^{2*} \end{bmatrix} \begin{bmatrix} \Phi_{\Gamma_1}^{1} & \Phi_{\Gamma_2}^{2} \end{bmatrix} \\
&= \begin{bmatrix} \Phi_{\Gamma_1}^{1*}\Phi_{\Gamma_1}^{1} & \Phi_{\Gamma_1}^{1*}\Phi_{\Gamma_2}^{2} \\ \Phi_{\Gamma_2}^{2*}\Phi_{\Gamma_1}^{1} & \Phi_{\Gamma_2}^{2*}\Phi_{\Gamma_2}^{2} \end{bmatrix} \\
&= \begin{bmatrix} I & \Phi_{\Gamma_1}^{1*}\Phi_{\Gamma_2}^{2} \\ \Phi_{\Gamma_2}^{2*}\Phi_{\Gamma_1}^{1} & I \end{bmatrix}.
\end{aligned}$$

   Therefore, $M = \Phi_{\Gamma_1}^{1*}\Phi_{\Gamma_2}^{2}$.

   (b) Since the maximum inner product between any column from Ψ^1 and any column from Ψ^2 is at most $3/\sqrt{n}$, the maximum possible value of an entry in M is $3/\sqrt{n}$.

   (c) Since $\Phi_\Gamma^*\Phi_\Gamma = I + G$ and $\|G\| \leq \|M\|$, $\|M\| < 1$ is a sufficient condition that $\Phi_\Gamma^*\Phi_\Gamma$ is invertible. Since M is a $|\Gamma_1| \times |\Gamma_2|$ matrix and its maximum possible value is $3/\sqrt{n}$,

$$\begin{aligned}
\|M\| &\leq \sqrt{\sum_{j,k} |M_{j,k}|^2} \\
&\leq \sqrt{|\Gamma_1| \cdot |\Gamma_2|} \cdot \max_{j,k}\{|M_{j,k}|\} \\
&\leq \sqrt{|\Gamma_1| \cdot |\Gamma_2|} \cdot 3/\sqrt{n}.
\end{aligned}$$

Therefore,

$$|\Gamma_1| \cdot |\Gamma_2| < n/9$$

is a sufficient condition that $\Phi_\Gamma^* \Phi_\Gamma$ is invertible.

(d) No $x \in \mathbb{R}^n$ can simultaneously have $\Psi^{1*}x$ non-zero only on Γ_1 and $\Psi^{2*}x$ non-zero only on Γ_2 when

$$\underline{|\Gamma_1| \cdot |\Gamma_2|} < \underline{n/9}.$$

The proof is shown as follows. Suppose there exists $x \in \mathbb{R}^n$ that simultaneously has $\Psi^{1*}x$ non-zero only on Γ_1 and $\Psi^{2*}x$ non-zero only on Γ_2, and $|\Gamma_1| \cdot |\Gamma_2| < n/9$. Let $f^1 = \Psi^{1*}x$ and $f^2 = \Psi^{2*}x$. Since Ψ^1 and Ψ^2 are orthobases,

$$\Psi^1 f^1 - \Psi^2 f^2 = x - x = 0.$$

Since f^1 and f^2 are supported on Γ_1 and Γ_2 respectively, the above equality can be formulated as

$$\begin{bmatrix} \Psi^1_{\Gamma_1} & \Psi^2_{\Gamma_2} \end{bmatrix} \begin{bmatrix} f^1_{\Gamma_1} \\ f^2_{\Gamma_2} \end{bmatrix} = 0.$$

Let $\Phi_\Gamma = \begin{bmatrix} \Psi^1_{\Gamma_1} & \Psi^2_{\Gamma_2} \end{bmatrix}$. Since $|\Gamma_1| \cdot |\Gamma_2| < n/9$ implies $\Phi_\Gamma^* \Phi_\Gamma$ is invertible, it can be derived that

$$\begin{aligned} f^1_{\Gamma_1} &= 0 \\ f^2_{\Gamma_2} &= 0, \end{aligned}$$

which contradicts f^1 and f^2 are supported on Γ_1 and Γ_2 respectively.

(e) It is impossible to find a vector $\beta \in \mathbb{R}^{2n}$ that has fewer non-zero terms than α and $\Phi\beta = f$ when

$$\underline{|\Gamma_1| + |\Gamma_2|} < \underline{\sqrt{n}/3}.$$

The proof is shown as follows. If there exists such a vector β, the number of the nonzero entries in both vectors α and β is less than $\sqrt{n}/3$ and they satisfy

$$\Phi\alpha - \Phi\beta = f - f = 0.$$

Let $\gamma = \alpha - \beta$. Then $\Phi\gamma = 0$ and the number of the nonzero entries in γ is less than $2\sqrt{n}/3$. Decompose γ as

$$\gamma = \begin{bmatrix} \gamma^1 \\ \gamma^2 \end{bmatrix},$$

where γ^1 and γ^2 are supported on Λ_1 and Λ_2, respectively. Then $|\Lambda_1| + |\Lambda_2| < 2\sqrt{n}/3$. If $\gamma \neq 0$, according to the proof in (d), Λ_1 and Λ_2 should also satisfy $|\Lambda_1| \cdot |\Lambda_2| \geq n/9$. It is easy to check that these two inequalities are contradictory.