Homework#4: Short course on sparse recovery and compressed sensing

1. **Answer:** Three lectures were given today. The first lecture focuses on another mathematical fundamental of compressive sensing — random matrices are restricted isometries. By putting together a bunch of easy results from probability theory, it is proved that an M by N iid Gaussian random matrix is a restricted isometry of size $2S$ with extraordinarily high probability when M is on the order of $S\log(N/S)$. The second lecture introduces some results on low rank recovery. Matrix completion aims to complete a matrix with observations of only some of its entries. It is shown that those blanks can be filled if the matrix is low rank and its singular vectors are incoherent. A more general problem of recovering a low rank matrix from a set of linear measurements is also considered. Finally, it is demonstrated that the problem of solving bilinear equations can be formulated to low rank recovery. The last lecture introduces a dynamical framework for sparse recovery, which is a recent research of Prof. Romberg and his collaborators. First, it talks about fast updating of solutions of ℓ_1 optimization programs in various scenarios, including the underlying signal changes slightly, add and remove measurements, the weights change, and have streaming measurements for an evolving signal. Second, it also introduces systems of nonlinear differential equations that solve ℓ_1 and related optimization programs, and it is implemented as continuous-time neural nets.

2. **Answer:** First, we prove that for a fixed $x \in \mathbb{R}^N$, the inequality

$$\mathrm{E}\|\Phi x\|_2^2 = \|x\|_2^2$$

holds. This can be proved by

$$\begin{aligned}\|\Phi x\|_2^2 &= \sum_{m=1}^{M}\left(\sum_{n=1}^{N}\Phi(m,n)x_n\right)^2 \\ &= \sum_{m=1}^{M}\left(\sum_{n=1}^{N}\Phi(m,n)^2 x_n^2 + \sum_{i\neq j}\Phi(m,i)\Phi(m,j)x_i x_j\right) \\ &= \|x\|_2^2 + \sum_{m=1}^{M}\sum_{i\neq j}\Phi(m,i)\Phi(m,j)x_i x_j\end{aligned}$$

and $\mathrm{E}(\Phi(m,i)\Phi(m,j)) = 0$ when $i \neq j$.

Next, we prove that for a fixed $x \in \mathbb{R}^N$, the inequalities

$$(1-\delta)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1+\delta)\|x\|_2^2$$

holds with probability exceeding $1 - 2\mathrm{e}^{-\delta^2 M/2}$. Define $z_{m,i,j} = \Phi(m,i)\Phi(m,j)x_i x_j$. It is easy to verify that $\{z_{m,i,j}\}_{i\neq j}$ are independent variables with $\mathrm{E}z_{m,i,j} = 0$ and $|z_{m,i,j}| =$

$|x_i x_j|/M$. According to the Hoeffding inequality,

$$\begin{aligned}
\mathrm{P}\left\{\left|\|\Phi x\|_2^2 - \|x\|_2^2\right| \geq \delta\|x\|_2^2\right\} &= \mathrm{P}\left\{\left|\sum_m \sum_{i\neq j} z_{m,i,j}\right| \geq \delta\|x\|_2^2\right\} \\
&\leq 2\exp\left(-\frac{\delta^2 M^2 \|x\|_2^4}{2\sum_{m=1}^M \sum_{i\neq j} x_i^2 x_j^2}\right) \\
&\leq 2\exp\left(-\frac{\delta^2 M^2 \|x\|_2^4}{2\sum_{m=1}^M \|x\|_2^4}\right) \\
&= 2\exp\left(-\frac{\delta^2 M}{2}\right),
\end{aligned}$$

which is exactly what we desired.

As for the rest of the proof, it is the same as that for the Gaussian case given in the lecture note, therefore it is omitted here.

3. **Answer:** For each case of random matrices, the simulation is repeated 1000 times to calculate the expected maximum singular values and the expected minimum singular values. The results are demonstrated in the following table.

M		50	75	100	200	300	1000
Gaussian	maximum	1.9340	1.7634	1.6642	1.4712	1.3852	1.2115
	minimum	0.0134	0.2058	0.3160	0.5195	0.6090	0.7863
Bernoulli	maximum	1.9064	1.7428	1.6466	1.4590	1.3751	1.2067
	minimum	0.0135	0.2088	0.3197	0.5247	0.6138	0.7901