Iterative Reconstruction of Bandlimited Graph Signals

Xiaohan Wang, Pengfei Liu, and Yuantao Gu

Abstract

Signal processing on graph is attracting more and more attention. For a graph signal in the low-frequency space, the missing data on the vertices of graph can be reconstructed through the sampled data by exploiting the smoothness of graph signal. In this paper, two iterative methods are proposed to reconstruct bandlimited graph signal from sampled data. In each iteration, one of the proposed methods conducts a neighboring duplication operation, while the other reweights the sampled residual for different vertices. Both the methods are proved to converge to the original signal under certain conditions. The proposed methods lead to a significantly faster convergence compared with the baseline method. Experiment results of synthetic graph signal and the real world data demonstrate the effectiveness of the reconstruction methods.

Index Terms

Graph signal, sampling, iterative reconstruction, frame theory

I. INTRODUCTION

A. Signal Processing on Graph

In recent years, the increasing demands for signal and information processing in irregular domains have resulted in the emerging field of signal processing on graphs [1], [2]. Bringing a new perspective for analysing data associated with graphs, graph signal processing has found applications in sensor networks [22], image processing [23], semi-supervised learning [36] and recommendation systems [5].

An undirected graph is denoted as $G(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the vertex set with vertex count $|\mathcal{V}| = N$ and $\mathcal{E}$ denotes the edge set. If one real value is associated with each of the vertices, then these values are collectively referred as a graph signal. A graph signal can also be regarded as a function $f : \mathcal{V} \rightarrow \mathbb{R}$.

There has been lots of research on graph signal related problems, including graph filtering [32], [33] graph wavelets [34], [35], [18], [27], uncertainty principle [25], multiresolution transforms [30], [29],
graph signal compression [24], graph signal sampling [4], [3], parametric dictionary learning [26] graph topology learning [31] and graph signal coarsening [28].

B. Problem Description and Related Works

Smooth signals or approximately smooth signals over graph are common in practical applications, especially for those cases in which the graph topologies are constructed to enforce the smoothness property of signals [31]. Exploiting the smoothness of a graph signal, it may be reconstructed through its values on only part of the vertices, i.e. samples of the graph signal.

In this work, we study the problem of reconstructing a bandlimited graph signal from known samples. The smooth signal is supposed to be with a low-frequency space. Iterative methods are proposed to recover its missing entries from the known sampled data.

There have been some theoretical analysis on the sampling and reconstruction of bandlimited graph signals [15], [16], [17]. Some existing works focus on the theoretical conditions for the exact reconstruction of bandlimited signals. The relationships between the sampling sets of unique reconstruction and the cutoff frequency of bandlimited signal space are established for normalized Laplacian and combinatorial Laplacian in [15] and [17], respectively. Recently, a necessary and sufficient condition of exact reconstruction is given in [3]. To reconstruct bandlimited graph signals from sampled values, several methods have been proposed. In [5] a least square approach is proposed to solve this problem. Furthermore, an iterative reconstruction method is proposed in [4], and a tradeoff between smoothness and data-fitting is introduced for real world applications.

The problem of signal reconstruction is closely related to the frame theory, which is also involved in other analysis of graph signals, especially in wavelet or vertex-frequency analysis on graphs [18]. Based on windowed graph Fourier transform and vertex-frequency analysis, windowed graph Fourier frames are studied in [20]. A spectrum-adapted tight vertex-frequency frame is proposed in [19] via translation on the graph. These works focus on vertex-frequency frames whose elements make up over-representation dictionaries, while in the reconstruction problem the frames are always composed by elements centering at the vertices in the sampling sets.

C. Contributions

In this paper, to improve the convergence rate of bandlimited graph signal reconstruction, iterative neighbor set reconstruction (INSR) and iterative adaptive weights reconstruction (IAWR) are proposed to reconstruct graph signals in bandlimited signal spaces. Both INSR and IAWR are theoretically proved
to reconstruct the original signal under certain conditions. The correspondence between graph signal sampling and time-domain irregular sampling is analyzed comprehensively, which will be helpful to future works on the sampling and reconstruction of graph signals. Experiments show that INSR and IAWR converge significantly faster than existing methods. Besides, experiments on several topics including sampling geometry and robustness are conducted.

The rest of this paper is organized as follows. In Section II, some preliminaries are introduced. In Section III and IV, INSR and IAWR are proposed and their convergence are analyzed, respectively. Section V includes some discussion on the proposed method and Section VI presents some numerical experiments.

II. PRELIMINARIES

A. Laplacian-based Graph Signal Processing

Graph Laplacians are extensively used in spectral graph theory [21] and signal processing on graphs [1] since they can reveal many properties of a graph. Two widely used types of graph Laplacians are combinatorial Laplacian (also referred as unnormalized Laplacian, or simply as Laplacian) and normalized Laplacian. The combinatorial Laplacian is defined as

$$L = D - A,$$

where $A$ is the adjacency matrix of the graph and $D$ is the diagonal degree matrix with diagonal elements as the degrees of the corresponding vertices. The normalized Laplacian is defined as

$$\mathcal{L} = D^{-\frac{1}{2}}LD^{-\frac{1}{2}}.$$

Both $L$ and $\mathcal{L}$ are operators on the space of graph signals over $G$,

$$(Lf)(u) = \sum_{v, v \sim u} (f(u) - f(v)),$$

$$(\mathcal{L}f)(u) = \frac{1}{\sqrt{d(u)}} \sum_{v, v \sim u} \left( \frac{f(u)}{\sqrt{d(u)}} - \frac{f(v)}{\sqrt{d(v)}} \right),$$

where $v \sim u$ means there is an edge in $G$ between vertices $v$ and $u$, and $d(v)$ denotes the degree of vertex $v$. Therefore, the Laplacian operators can be viewed as a kind of difference operation between the central vertices and their neighbors.

The Laplacian is a real symmetric matrix, thus all the eigenvalues are nonnegative. Suppose $\{\lambda_k\}$ are the eigenvalues, and $\{u_k\}$ are the corresponding eigenvectors. Because of the property of Laplacian,
the eigenvectors associated with small eigenvalues have similar values for connected vertices, while the eigenvectors associated with large eigenvalues vary fast on the graph. The graph Fourier transform is defined as the expansion of a graph signal \( f \) in terms of \( \{u_k\} \), as,

\[
\hat{f}(\lambda_k) = \langle f, u_k \rangle = \sum_{i=1}^{N} f(i) u_k(i),
\]

where \( f(i) \) denotes the entry of \( f \) associated with vertex \( i \). Similar with classical Fourier analysis, eigenvalues \( \{\lambda_k\} \) are regarded as frequencies of the graph, and \( \hat{f}(\lambda_k) \) is regarded as the frequency component corresponding to \( \lambda_k \). Therefore, the frequency components with smaller eigenvalues can be called low-frequency part, and those with larger eigenvalues is the high-frequency part.

For a graph signal \( f \in \mathbb{R}^N \) on a graph \( G(V,E) \), \( f \) is called \( \omega \)-bandlimited if the spectral support of \( f \) is within \([0, \omega]\). That is, the frequency components corresponding to eigenvalues larger than \( \omega \) are all zero. The subspace of \( \omega \)-bandlimited signals on graph \( G \) form a Hilbert space called Paley-Wiener space, and is denoted as \( PW_\omega(G) \) [15].

In this paper, we consider the sampling and reconstruction problem. Suppose that for a bandlimited graph signal \( f \in PW_\omega(G) \), only signal values \( \{f(u)\}_{u \in S} \) on the sampled set \( S \subseteq V \) are known, the problem is to obtain the original signal \( f \) from the sampled signal values.

### B. Frame and Signal Reconstruction

The problem of signal sampling and reconstruction is closely related to the frame theory, as introduced in the following.

**Definition 1:** A family of elements \( \{f_i\}_{i \in I} \) is a frame for a Hilbert space \( \mathcal{H} \), if there exist constants \( 0 < A \leq B \) such that

\[
A \|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{H},
\]

where \( A \) and \( B \) are called frame bounds.

**Definition 2:** For a frame \( \{f_i\}_{i \in I} \), the frame operator \( S : \mathcal{H} \rightarrow \mathcal{H} \) is defined as

\[
Sf = \sum_{i \in I} \langle f, f_i \rangle f_i.
\]

Furthermore, \( S \) is invertible and satisfies \( AI \leq S \leq BI \), where \( I \) denotes the identity operator.

If a scalar \( \lambda \) satisfying \( \|I - \lambda S\| < 1 \), using the series expansion of \( S^{-1} \) one has

\[
f = S^{-1}f = \lambda \sum_{j=0}^{\infty} (I - \lambda S)^j Sf.
\]
By defining
\[ f^{(k)} = \lambda \sum_{j=0}^{k} (I - \lambda S)^j Sf, \]
the following iterative form is obtained,
\[ f^{(k+1)} = \lambda Sf + (I - \lambda S)f^{(k)} = f^{(k)} + \lambda S(f - f^{(k)}), \]
with the error bound satisfying
\[ \|f^{(k)} - f\| \leq \|I - \lambda S\|^k \|f^{(0)} - f\|. \]

The parameter \( \lambda \) affects the convergence rate. If \( \lambda \) is chosen as \( \lambda = \frac{1}{B} \), then \( \|I - \lambda S\| \leq \frac{B-A}{B} < 1 \), and the error bound of iteration (1) will shrink with the exponential of \( \frac{B-A}{B} \). A better choice of \( \lambda \) is \( \lambda = \frac{2}{A+B} \), then \( \|I - \lambda S\| \leq \frac{B-A}{B+A} \), which can lead to a faster convergence rate [10].

C. Previous Works on Bandlimited Graph Signal Sampling and Reconstruction

There have existed some useful theoretical results on the problem of bandlimited graph signal sampling and reconstruction. A concept of uniqueness set is firstly introduced in [15].

**Definition 3**: [15] A set of vertices \( \mathcal{U} \subseteq \mathcal{V}(\mathcal{G}) \) is a uniqueness set for space \( PW_\omega(\mathcal{G}) \) if \( \forall f, g \in PW_\omega(\mathcal{G}), f_\mathcal{U} = g_\mathcal{U} \) implies \( f = g \), where \( f_\mathcal{U} \) denotes the restriction of \( f \) on \( \mathcal{U} \).

In other words, \( \forall f \in PW_\omega(\mathcal{G}), f \) is uniquely determined by its values on the uniqueness set \( \mathcal{U} \). It means that \( f \) can be exact recovery if the sampling set contains the uniqueness set. Readers may refer to [15], [5] and [3] for more details of the conditions for uniqueness set.

It is proved in [15] that a set of graph signals related to the uniqueness set becomes a frame for \( PW_\omega(\mathcal{G}) \), which is a quite important foundation of our work as the following theorem.

**Theorem 1**: [15] If the sampling set \( \mathcal{S} \) is a uniqueness set for \( PW_\omega(\mathcal{G}) \), then \( \{P_\omega(\delta_u)\}_{u \in \mathcal{S}} \) is a frame for \( PW_\omega(\mathcal{G}) \), where \( P_\omega(\cdot) \) is the projection operator onto \( PW_\omega(\mathcal{G}) \), and \( \delta_u \in \mathbb{R}^N \) is a \( \delta \)-function whose entries satisfying
\[ \delta_u(v) = \begin{cases} 1, & v = u; \\ 0, & v \neq u. \end{cases} \]

A method named iterative least square reconstruction (ILSR) is proposed to reconstruct bandlimited graph signals iteratively [4]. To be consistent with our work, it is equivalently rewritten as follows.

**Theorem 2**: [4] If the sampling set \( \mathcal{S} \) is a uniqueness set for \( PW_\omega(\mathcal{G}) \), then the original signal \( f \) can be reconstructed using the sampled data \( \{f(u)\}_{u \in \mathcal{S}} \) by the method ILSR method in Table I.
TABLE I
ITERATIVE LEAST SQUARE RECONSTRUCTION.

<table>
<thead>
<tr>
<th>Input:</th>
<th>Graph ( G ), cutoff frequency ( \omega ), sampling set ( S ), sampled data ( { f(u) }_{u \in S} );</th>
</tr>
</thead>
<tbody>
<tr>
<td>Output:</td>
<td>Interpolated signal ( f );</td>
</tr>
</tbody>
</table>

Initialization:

\[ f^{(0)} = P_{\omega} \left( \sum_{u \in S} f(u) \delta_u \right); \]

Loop:

\[ f^{(k+1)} = f^{(k)} + P_{\omega} \left( \sum_{u \in S} (f(u) - f^{(k)}(u)) \delta_u \right); \]

Until: The stop condition is satisfied.

Because of Theorem 1, the frame operator associated with the frame \( \{ P_{\omega}(\delta_u) \}_{u \in S} \) is

\[ Sf = \sum_{u \in S} \langle f, P_{\omega}(\delta_u) \rangle P_{\omega}(\delta_u). \]

For \( f \in PW_\omega(\mathcal{G}) \), we have \( P_{\omega}(f) = f \), therefore

\[ \langle f, P_{\omega}(\delta_u) \rangle = \langle P_{\omega}(f), \delta_u \rangle = \langle f, \delta_u \rangle = f(u). \]

Then ILSR can also be obtained when \( \lambda \) is chosen as \( \lambda = 1 \) in the iteration (1). In fact in [4] this method is not derived from the frame theory, however we can rewrite the method into a frame-based form.

It is necessary to note that some of the above theoretical results are based on the normalized Laplacian matrix. However, similar results can be easily obtained for Laplacian matrix. In our work, the Laplacian matrix is mainly used.

III. ITERATIVE NEIGHBOR SET RECONSTRUCTION

In this section, to improve the convergence rate of ILSR, a reconstruction method named iterative neighbor set reconstruction (INSR) is proposed and its convergence is theoretically proved.

A. Preparations

Given the sampling set, a division of the graph is needed. Assume that the whole vertex set \( V \) is divided into disjoint neighbor sets \( \{ \mathcal{N}(u) \}_{u \in S} \) of the sampled vertices in \( S \). For each \( u \in S \), \( u \in \mathcal{N}(u) \), and the
subgraph which is composed by the vertices in $\mathcal{N}(u)$ and the edges between them in $\mathcal{G}$ is connected. Besides, $\{\mathcal{N}(u)\}_{u \in S}$ satisfies

$$\mathcal{N}(u) \cap \mathcal{N}(v) = \emptyset, u \neq v,$$

and

$$\bigcup_{u \in S} \mathcal{N}(u) = \mathcal{V}(\mathcal{G}).$$

In Definition 4 and Definition 5 two parameters are defined to describe the property of neighbor set $\mathcal{N}(u)$, which is useful in following analysis.

**Definition 4:** Denote

$$\mathcal{T}(u) = \text{SPT}(\mathcal{G}_{\mathcal{N}(u)})$$

as the shortest-path tree of $\mathcal{G}_{\mathcal{N}(u)}$ rooted at $u$, where $\mathcal{G}_{\mathcal{N}(u)}$ is the subgraph of $\mathcal{G}$ which is composed by vertices in $\mathcal{N}(u)$ and edges between them in $\mathcal{G}$. For $v$ connected to $u$ in $\mathcal{T}(u)$, $\mathcal{T}_u(v)$ is the subtree which $v$ belongs to when $u$ and its associate edges are deleted in $\mathcal{T}(u)$. The maximal multiple number of $\mathcal{N}(u)$ is defined as

$$K(u) = \max_{(u,v) \in \mathcal{E}(\mathcal{T}(u))} |\mathcal{T}_u(v)|,$$

where $\mathcal{E}(\mathcal{T}(u))$ is the edge set of graph $\mathcal{T}(u)$, and $|\mathcal{T}_u(v)|$ is the number of vertices of $\mathcal{T}_u(v)$.

**Remark 1:** By the definition of $K(u)$, it is easy to see that

$$K(u) \leq |\mathcal{N}(u)| - d_{\mathcal{N}(u)}(u) \leq |\mathcal{N}(u)| - 1,$$

(2)
where \(|\mathcal{N}(u)|\) is the cardinal number of set \(\mathcal{N}(u)\) and \(d_{\mathcal{N}(u)}(u)\) is the degree of \(u\) in the subgraph \(\mathcal{G}_{\mathcal{N}(u)}\).

For simplicity, one may denote

\[
\tilde{K}(u) = |\mathcal{N}(u)| - d_{\mathcal{N}(u)}(u). \tag{3}
\]

The definitions above are intuitively illustrated in Fig. 1.

Definition 5: The radius of \(\mathcal{N}(u)\) is the maximal distance from \(u\) for the vertices in \(\mathcal{N}(u)\), which is denoted as

\[
R(u) = \max_{v \in \mathcal{N}(u)} d(v, u).
\]

B. Iterative Neighbor Set Reconstruction

Based on the parameters we have the following lemma.

Lemma 1: Assume that \(A\) is an operator defined by

\[
Af = \mathcal{P}_\omega \left( \sum_{u \in S} (f, \delta_u) \delta_{\mathcal{N}(u)} \right),
\]

where \(\delta_{\mathcal{N}(u)}\) denotes the \(\delta\)-function of set \(\mathcal{N}(u)\) with entries

\[
\delta_{\mathcal{N}(u)}(v) = \begin{cases} 
1, & v \in \mathcal{N}(u); \\
0, & v \notin \mathcal{N}(u).
\end{cases}
\]

then \(\forall f \in \text{PW}_\omega(\mathcal{G})\),

\[
\|f - Af\| \leq \sqrt{Q_{\max}} \|f\|,
\]

where

\[
Q_{\max} = \max_{u \in S} K(u)R(u).
\]

The proof of Lemma 1 is postponed to VIII-A.

Lemma 1 implies that the operator \(I - A\) is bounded in \(\text{PW}_\omega(\mathcal{G})\). As a consequence, for \(\omega\) small enough, it leads to \(|I - A| < 1\), which is a contraction mapping. Considering \(\langle f, \delta_u \rangle = f(u)\), the sampled data \(\{f(u)\}_{u \in S}\) can be used to reconstruct the original signal as shown in Proposition 1.

Proposition 1: Suppose that the original graph signal \(f \in \text{PW}_\omega(\mathcal{G})\). If

\[
\sqrt{Q_{\max}} \omega = \gamma < 1,
\]

then \(f\) can be reconstructed by the sampled data \(\{f(u)\}_{u \in S}\) through the INSR method in Table II, with the error bound satisfying

\[
\|f^{(k)} - f\| \leq \gamma^k \|f^{(0)} - f\|.
\]
TABLE II
ITERATIVE NEIGHBOR SET RECONSTRUCTION.

Input: Graph $\mathcal{G}$, cutoff frequency $\omega$, sampling set $\mathcal{S}$, neighbor sets $\{\mathcal{N}(u)\}_{u \in \mathcal{S}}$, sampled data $\{f(u)\}_{u \in \mathcal{S}}$;

Output: Interpolated signal $f$;

Initialization:
\[
f(0) = \mathcal{P}_\omega \left( \sum_{u \in \mathcal{S}} f(u) \delta_{\mathcal{N}(u)} \right);
\]

Loop:
\[
f^{(k+1)} = f^{(k)} + \mathcal{P}_\omega \left( \sum_{u \in \mathcal{S}} (f(u) - f^{(k)}(u)) \delta_{\mathcal{N}(u)} \right);
\]

Until: The stop condition is satisfied.

The proof of Proposition 1 is in VIII-B. Proposition 1 implies that the signal $f$ can be reconstructed with $\{f(u)\}_{u \in \mathcal{S}}$ by copying the residual of sampled vertices to other vertices in $\mathcal{N}(u)$ and $\omega$-bandlimited projection iteratively.

Remark 2: If the sampling set $\mathcal{S} = \mathcal{V}$, then $|\mathcal{N}(u)| = 1$, $K(u) = 0$ and $Q_{\text{max}} = 0$. Therefore according to Proposition 1, any $f$ satisfying $\omega < \infty$ can be reconstructed, which is a natural result.

Corollary 1: In Lemma 1 and Proposition 1 $Q_{\text{max}}$ can be replaced by $\tilde{Q}_{\text{max}}$, which is defined as
\[
\tilde{Q}_{\text{max}} = \max_{u \in \mathcal{S}} \tilde{K}(u) R(u).
\]

According to (2), for any $u \in \mathcal{S}$ we have $\tilde{K}(u) \geq K(u)$, and then $\tilde{Q}_{\text{max}} \geq Q_{\text{max}}$. In fact, $K(u)$ is not very easy to obtain for each given subgraph $\mathcal{G}_{\mathcal{N}(u)}$. However, $\tilde{K}(u)$ is very convenient to get and $\tilde{Q}_{\text{max}}$ is a more practical choice, even though the bound is not as accurate.

C. A Result on the Frame

Based on Lemma 1, the $\omega$-bandlimited projection of the $\delta$-function can be proved to be a frame and its frame bounds can be estimated.

Proposition 2: If $\omega$ satisfies
\[
\sqrt{Q_{\text{max}} \omega} = \gamma < 1,
\]
then $\{\mathcal{P}_\omega(\delta_{\mathcal{N}(u)})\}_{u \in \mathcal{S}}$ is a frame in $PW_\omega(\mathcal{G})$ with bounds $(1 - \gamma)^2$ and $N_{\text{max}}$, and $\{\mathcal{P}_\omega(\delta_u)\}_{u \in \mathcal{S}}$ is a frame in $PW_\omega(\mathcal{G})$ with bounds $(1 - \gamma)^2/N_{\text{max}}$ and 1.
The proof of Proposition 2 is in VIII-C. Proposition 2 implies an implicit relationship between frame and sampling reconstruction. In fact the operator $A$ is not a standard form of a frame operator, since two different families of signals $\{P\omega(\delta_u)\}_{u \in S}$ and $\{P\omega(\delta_{N(u)})\}_{u \in S}$ are involved. However, under the same condition with the contraction of operator $I - A$, both the two families of signals can be proved to be frames, and either of them can be used to reconstruct the original signal by the corresponding frame operator.

Theorem 3.2 of [17] also implies that $\{P\omega(\delta_u)\}_{u \in S}$ is a frame in $PW_\omega(G)$. However, we follow different ways under different assumptions and obtain different bounds estimations. Since Proposition 2 is a byproduct of our analysis, it is not deeply studied here. Further work can be done to improve the frame bounds estimations.

D. The One-hop Sampling Set

From the above analysis, the condition is determined by both $K(u)$ and $R(u)$, which are two different quantities to describe how the vertices in $N(u)$ are concentrated around $u$. However, both the quantities can be small simultaneously in some special cases. Corollary 2 provides a corresponding result for a more special case. If the sampling set is chosen carefully, the condition in the following Corollary 2 is not very difficult to satisfy.

**Corollary 2:** If the neighbor set of $\forall u \in S$ satisfies

$$\max_{v \in N(u)} d(u, v) \leq 1,$$

the sufficient condition of $\omega$ in Proposition 1 can be refined as $\omega < 1$ and $\gamma$ can be refined as $\gamma = \sqrt{\omega}$.

The condition (4) means that all the vertices except $u$ in $N(u)$ are connected to $u$. It implies that $d_{N(u)}(u) = |N(u)| - 1$ and then $\tilde{K}(u) \leq 1$, according to (3). Besides, it is obvious to see $R(u) \leq 1$. Therefore, $Q_{\text{max}} = 1$ and Corollary 2 is obtained.

A sampling set with a smaller $Q_{\text{max}}$ can expand the scope of bandlimited signal which can be guaranteed to reconstruct. Besides, for a given $\omega$, a smaller $Q_{\text{max}}$ leads to a smaller $\gamma$ and a better error bound of convergence. Therefore, when the graph topology is given, it is necessary to find a proper sampling set $S$ and the corresponding vertex division $\{N(u)\}_{u \in S}$ which makes the quantity $Q_{\text{max}}$ as small as possible.

However, the simplest case is good enough, in which the condition (4) is satisfied. It is a rather economical choice of sampling set when there is no restriction on the number or location of the sampling vertices because both $K(u)$ and $R(u)$ will be small simultaneously. A greedy method described in Table III can be conducted to get the one-hop sampling set, which satisfies the condition of (4).
TABLE III
A GREEDY METHOD FOR A ONE-HOP SAMPLING SET.

Input: Graph $G(V, E)$;
Output: One-hop sampling set $S$, neighbor sets $\{N(u)\}_{u \in S}$;
Initialization: $S = \emptyset$;
Loop:
1) Find the largest-degree vertex $u = \arg \max_{v \in V} d_G(v)$;
2) Add $u$ into the sampling set $S = S \cup \{u\}$;
3) The one-hop neighbor set $N(u) = \{u\} \cup \{v \in V | (u, v) \in E\}$;
4) Remove the edges $E = E \setminus \{(p, q)| p \in N(u), q \in V\}$;
5) Remove the vertices $V = V \setminus N(u)$ and $G = G(V, E)$;
Until: $V = \emptyset$.

IV. ITERATIVE ADAPTIVE WEIGHTS RECONSTRUCTION

Another iterative reconstruction method called iterative adaptive weights reconstruction (IAWR) is introduced in this section. It is also based on the frame theory. A lemma is firstly proposed to prove that the weighted signals $\{w(u)P_\omega(\delta_u)\}_{u \in S}$ is a frame when the weights are chosen as $w(u) = \sqrt{|N(u)|}$ for all $u \in S$.

Lemma 2: If $\omega$ satisfies
$$\sqrt{Q_{\max}} \omega = \gamma < 1,$$
then
$$(1 - \gamma) \parallel f \parallel^2 \leq \sum_{u \in S} |N(u)| \cdot |f(u)|^2 \leq (1 + \gamma)^2 \parallel f \parallel^2$$
for $\forall f \in PW_\omega(G)$. It means that $\{\sqrt{|N(u)|}P_\omega(\delta_u)\}_{u \in S}$ is a frame in $PW_\omega(G)$ with bounds $(1 - \gamma)^2$ and $(1 + \gamma)^2$.

The proof of Lemma 2 is postponed to VIII-D.

Proposition 2 shows that $\{P_\omega(\delta_u)\}_{u \in S}$ is a frame with bounds $(1 - \gamma)^2/N_{\max}$ and 1. Because of the assumption of finite graph, $\{\sqrt{|N(u)|}P_\omega(\delta_u)\}_{u \in S}$ is naturally a frame following Proposition 2 and an estimation of frame bounds can be obtained immediately as $\frac{N_{\min}}{N_{\max}}(1 - \gamma)^2$ and $N_{\max}$, where $N_{\min} = \min_{u \in S}|N(u)|$. 

July 24, 2014 DRAFT
TABLE IV
ITERATIVE ADAPTIVE WEIGHTS RECONSTRUCTION.

| Input: | Graph $G$, cutoff frequency $\omega$, sampling set $S$, neighbor sets $\{N(u)\}_{u \in S}$, sampled data $\{f(u)\}_{u \in S}$; |
| Output: | Interpolated signal $f$; |

Initialization:

$$f^{(0)} = \frac{1}{1 + \gamma^2} P_\omega \left( \sum_{u \in S} |N(u)| f(u) \delta_u \right);$$

Loop:

$$f^{(k+1)} = f^{(k)} + \frac{1}{1 + \gamma^2} P_\omega \left( \sum_{u \in S} |N(u)| (f(u) - f^{(k)}(u)) \delta_u \right);$$

Until: The stop condition is satisfied.

We will compare the two estimations of the frame bounds. Obviously Lemma 2 gives a sharper lower bound than that derived from Proposition 2, because of the inequality $N_{\min} \leq N_{\max}$. Since $\gamma < 1$, if $N_{\max} \geq 4$ then $N_{\max} > (1 + \gamma)^2$ and the upper bound by Lemma 2 is also sharper. If $N_{\max} \leq 3$ the relationship between the two estimation depends on $\omega$. A more precise condition is when $\omega \leq (\sqrt{N_{\max}} - 1)^2/Q_{\max}$, the upper bound of Lemma 2 is better.

To make the expression consistent and concise, we use the bound estimation provided by Lemma 2 hereinafter.

Adaptive weights method is introduced in Proposition 3 and its convergence is proved.

**Proposition 3:** Suppose that the original graph signal $f \in PW_\omega(G)$. If $\omega$ satisfies

$$\sqrt{Q_{\max}} \omega = \gamma < 1,$$

the original signal $f$ can be reconstructed by the sample data $\{f(u)\}_{u \in S}$ through the IAWR method in Table IV, with the error bound satisfying

$$\|f^{(k)} - f\| \leq \left( \frac{2\gamma}{1 + \gamma^2} \right)^k \|f^{(0)} - f\|.$$

The proof of Proposition 3 is postponed to VIII-E. The idea of adaptive weights method is to attach different weights to sampled vertices. The weights for vertex $u$ is larger if its neighboring set $N(u)$ has more vertices, in other words, the vertex $u$ is more isolated or the region around $u$ has a lower sampling
density. On the contrary, if the sampled vertices in a region are very dense, less importance is allocated to them.

**Remark 3:** Since $\frac{2\gamma}{1+\gamma^2} > \gamma$ when $\gamma < 1$, the theoretical guarantee of INSR converges faster than that of IAWR.

**Remark 4:** Similar to Proposition 1, $Q_{\text{max}}$ can also be replaced by $\tilde{Q}_{\text{max}}$ in Lemma 2 and Proposition 3, which is more practical to obtain.

In fact, Proposition 3 is a natural generalization of existed frame-based reconstruction method when the adaptive weighted signal $\{\sqrt{|N(u)|}P_\omega(\delta_u)\}_{u \in S}$ is proved to be a frame in Lemma 2.

V. DISCUSSION

A. Relationship with Time Domain Results

Bandlimited signal sampling and reconstruction on graph has close relationship with irregular sampling [6], [13] or nonuniform sampling [7] in the time domain, which sheds light on the analysis of graph signal. There have existed several iterative reconstruction methods and theoretically analysis of time-domain irregular sampling [14], [8], [9], [6], some of which are related to the frame theory [8], [6], [12].

By exploiting the similarities between time-domain irregular sampling and graph signal sampling, some results of this work have consistent formulation with the corresponding results in the time domain. The reconstruction methods also have correspondences in the time domain.
Results on time-domain irregular sampling show that \( \{ T_{t_i} \text{sinc} \}_{t_i \in S} \) is a frame in \( B^2_{\Omega} \) if the sampling set \( S \) satisfies some particular conditions, where \( T_{t_i}f(t) = f(t - t_i) \) denotes the translation of \( f(t) \), \( \text{sinc} \) denotes the sinc function whose bandwidth is \( \Omega \) and \( B^2_{\Omega} \) denotes the space of \( \Omega \)-bandlimited square integrable signal.

Correspondingly, for the graph signal \( f \in PW_{\omega}(G) \), \( \{ P_{\omega}(\delta_u) \} \) \( u \in S \) is a frame under some conditions. The result is consistent with that in the time domain. The correspondence between irregular sampling in the time domain and that on graph is illustrated in Fig. 2. In the graph signal sampling problem, \( \{ P_{\omega}(\delta_u) \} \) \( u \in S \) corresponds to the frame \( \{ T_{t_i} \text{sinc} \} \) \( t_i \in S \) in the time domain. The essence of the two problem is very similar and theoretical results on sampling and reconstruction of graph signal can be obtained enlightened by irregular sampling in the time domain.

The graph signal reconstruction methods ILSR, INSR and IAWR have correspondences in time-domain, which are Marvasti method [8], Voronoi method [9] and adaptive weights method [6], respectively.

The correspondence between time-domain irregular sampling and graph signal sampling is shown in Table V.

**B. Intuitive Explanation of the Three Methods**

The key procedures of each iteration of ILSR, INSR and IAWR are illustrated in Fig. 3. The major difference of the three iterative reconstruction methods is the way to deal with the residual of the sampled vertices. In each iteration the sampled residual \( \sum_{u \in S} (f(u) - f^{(k)}(u)) \delta_u \) is directly projected onto the \( \omega \)-bandlimited space \( PW_{\omega}(G) \) in ILSR method. For INSR method, the residuals of the sampled vertices are copied and assigned to the vertices in the corresponding neighboring sets and then the projection procedure is conducted. In adaptive weights method, the sampled residuals are multiplied by the adaptive weights \( |\mathcal{N}(u)| \) and then projected onto the low-frequency space. For each step the increment of INSR method or adaptive weights method is larger than that of ILSR method. It may partly explain why the proposed two methods converge faster than ILSR method.

Besides, in each iteration ILSR method and adaptive weights method use only local information for all the vertices, while data has to be transmitted from the sampled vertices to their neighbors in INSR method. As a result, the former two methods may be easier to be applied in potential distributed scenario.

**VI. EXPERIMENTS**

The Minnesota road graph [11] is chosen as the graph, which has 2640 vertices and 6604 edges. The bandlimited signal is randomly generated by cutting off the high-frequency components. For simplicity,
TABLE V
THE CORRESPONDENCE BETWEEN IRREGULAR SAMPLING IN THE TIME DOMAIN AND THAT ON GRAPH.

<table>
<thead>
<tr>
<th>Time Domain</th>
<th>Vertex Domain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Signal</td>
<td>$f(t)$</td>
</tr>
<tr>
<td>Cutoff frequency</td>
<td>$\Omega$</td>
</tr>
<tr>
<td>Low-frequency space</td>
<td>$B_{\Omega}^{2}$</td>
</tr>
<tr>
<td>Shifted sinc function</td>
<td>$T_{1,\text{sinc}_{\Omega}}$</td>
</tr>
<tr>
<td>Neighborhood</td>
<td>$[(t_{i-1} + t_{i})/2, (t_{i} + t_{i+1})/2)$</td>
</tr>
<tr>
<td>Neighbor indicator</td>
<td>$\mathbb{1}<em>{[(t</em>{i-1} + t_{i})/2, (t_{i} + t_{i+1})/2)}$</td>
</tr>
<tr>
<td>Adaptive weight</td>
<td>$\sqrt{(t_{i+1} - t_{i-1})/2}$</td>
</tr>
<tr>
<td>Reconstruction method</td>
<td>Marvasti Method</td>
</tr>
<tr>
<td>Reconstruction method</td>
<td>Voronoi Method</td>
</tr>
<tr>
<td>Reconstruction method</td>
<td>Adaptive Weights Method</td>
</tr>
</tbody>
</table>

a one-hop sampling set satisfying Corollary 2 is chosen as the sampling set if it is not specially claimed. The one-hop sampling set and the corresponding neighbouring sets are obtained by the greedy method mentioned in the end of subsection III-D. This sampling set has 872 vertices, which is about one third of all the vertices.

A. Convergence of Reconstruction Methods

1) INSR Method and Adaptive Weights Method: The convergence curves of ILSR, INSR and adaptive weights methods are illustrated in Fig. 4. Note that additional assignment operation is needed in each step of INSR method and multiplication operation is need in the adaptive weights method. However, it is obvious that the convergence rate of proposed methods is significantly improved compared with ILSR method. INSR method is better than adaptive weights method on the convergence rate.
Fig. 3. Illustration of the iterations of the three methods. Compared with ILSR, INSR needs to assign the residuals on the sampled vertices to their neighboring vertices, while IAWR needs to multiply the residuals on the sampled vertices by different weights.

Fig. 4. Convergence curves of ILSR, INSR and IAWR.

B. Sampling Geometry and Convergence Rate

1) Sample Sets with Different Quantities of $Q_{\text{max}}$: The choice of sampling set may affect the performance of convergence. This experiment verifies the analysis. Two different sampling sets are used to reconstruct the same bandlimited original signal. Both of the sampling sets have the same amount of vertices. The first sampling set is randomly chosen in the vertices, while the second sampling set is
the one-hop set satisfying Corollary 2. The convergence curves of the two sampling set using the three reconstruction methods are illustrated in Fig. 5. For all the three reconstruction methods, the convergence is faster by using the sampled data of the one-hop sampling set than the randomly chosen vertex set. It means that the sampling geometry has influence on the reconstruction.

2) Irregularity of Sampling Geometry: The irregularity of sampling geometry may also have influence on the convergence rate. On base of the one-hop sampling set, some vertices are added into the sampling set by two different ways, which will lead to an irregular sampling set and a relatively uniform one. One way to add vertices is to add the vertices densely around only several vertices, while the other way is to add vertices uniformly in the graph. Both of the two newly obtained sampling sets have the same amount of vertices. The convergence curves are shown in Fig. 6. For ILSR method, it can be seen that by uniformly adding vertices into the sampling set, the convergence will be faster and the convergence rate is not changed by adding vertices irregularly. However, since the curves almost coincide, both the two ways of adding vertices have little influence on the convergence for INSR method and adaptive weights method. It may imply that the two proposed methods have made full use of the sampled information in this case.

C. Actual and Priori Known Cutoff Frequencies

The cutoff frequency is a crucial quantity in the reconstruction of the bandlimited signal. For a bandlimited signal the cutoff frequency is known as a priori knowledge. However, the priori knowledge may be only an estimation from experience and may be inaccurate. In this case, the actual cutoff frequency
Fig. 6. Convergence curves of basic sampling set, irregularly and uniformly vertices added sampling sets.

Fig. 7. Convergence curves of three cases with different actual and priori known cutoff frequencies.

of the signal may be less than the known frequency. In this experiment, the effect on the imprecise knowledge of the cutoff frequency is investigated. For frequencies $\omega_1$ and $\omega_2$ satisfying $\omega_1 < \omega_2$, three cases are considered.

- The actual cutoff frequency is $\omega_1$ and the priori known frequency is also $\omega_1$;
- The actual cutoff frequency is $\omega_1$, while the priori known frequency is $\omega_2$;
- The actual cutoff frequency is $\omega_2$ and the priori known frequency is also $\omega_2$.

The convergence curves are illustrated in Fig. 7. For the first and second cases, although both the original signals are actually $\omega_1$-bandlimited, the reconstruction converges faster in the first case for a better priori
knowledge. Comparing the second and third cases, it can be seen that the convergence curves almost coincide, which means that although the original signal has a lower bandwidth, it will not provide any benefit if its smoothness is not priori known. The experiment results show that the convergence rate depends little on the actual cutoff frequency but depends more on the cutoff frequency the signal is regarded to have. If we have more accurate priori knowledge on the cutoff frequency, the reconstruction will be more efficient.

D. Robustness against Noise

1) Observation Noise: Suppose there is noise involved in the observation of sampled graph signal. This experiment focuses on the robustness to the observation noise of the three reconstruction methods. In this experiment the noise is generated as independent identical distributed Gaussian noise. As shown in Fig. 8, the root mean square error decreases as the SNR increases. The three methods have almost the same performance against observation noise.

2) Noise of Lowpass Filter: Since a projection onto the $\omega$-bandlimited space, which is actually a lowpass operation, is used in each iteration, the inaccuracy of the filter affect the reconstruction result. This experiment focuses on the robustness to the noise of lowpass filter. As illustrated in Fig. 9, it is natural to see that RMSE decreases as the SNR of lowpass filter increases. Among the three reconstruction methods, the adaptive weights method is the most robust to the noise of filter, followed by INSR method, and ILSR method is easier to be affected by the noise of filter.
3) Reconstruction of Approximated Bandlimited Signals: Real-world data is always not strictly bandlimited. However, most smooth signals over graph can be regarded as approximated bandlimited signals. In the experiment in Fig. 10, the three methods are used to reconstruct signals with different out-of-band energy. The RMSE will be larger for signals with more energy out of band. Besides, the three methods perform almost the same for approximated bandlimited signals.
VII. CONCLUSION

In this paper, the problem of graph signal reconstruction in low-frequency subspace is studied. Two methods called INSR and IAWR are proposed to iteratively reconstruct the missing data from the observed samples. Strict proofs of convergence and error bounds of INSR and IAWR are given. Experiments show that INSR performs beyond IAWR, and both the proposed methods converge significantly faster and are more robust than the existing ILSR method.

VIII. APPENDIX

A. Proof of Lemma 1

Proof: By the definition of $A$,

$$
\|f - Af\|^2 = \|P_\omega \left( f - \sum_{u \in S} (f, \delta_u) \delta_{N(u)}\right) \|^2
$$

$$
\leq \left\| \sum_{u \in S} (f_{N(u)} - f(u) \delta_{N(u)}) \right\|^2
$$

$$
= \sum_{u \in S} \left( \sum_{v \in N(u)} |f(v) - f(u)|^2 \right), \quad (5)
$$

where each entry of $f_{N(u)}$ is

$$
f_{N(u)}(v) = \begin{cases} 
  f(v), & v \in N(u); \\
  0, & v \notin N(u).
\end{cases}
$$

Since $N(u)$ is connected, there is a shortest path within $N(u)$ from $v$ to $u$, which is denoted as $v \sim v_1 \sim \cdots \sim v_k \sim u$. Then for $v \in N(u)$,

$$
|f(v) - f(u)|^2 \leq R(u) \left( |f(v) - f(v_1)|^2 + \cdots + |f(v_k) - f(u)|^2 \right), \quad (6)
$$

which is because any path is not longer than $R(u)$.

For each $v$ satisfying $(u, v) \in E(T(u))$, the path from any vertex in $T_u(v)$ to $u$ contains the edge $(u, v)$ and this edge is counted for $|T_u(v)|$ times. By the definition of $K(u)$, each edge in $N(u)$ is counted for no more than $K(u)$ times. Therefore,

$$
\sum_{v \in N(u)} |f(v) - f(u)|^2 \leq K(u) R(u) \sum_{p \neq q \in N(u)} |f(p) - f(q)|^2, \quad (7)
$$

July 24, 2014 DRAFT
By the assumption of $\omega$-bandlimited signal, the following inequality is established.

$$\sum_{p \sim q} |f(p) - f(q)|^2 = \sum_{p \in \mathcal{V}(G)} |f(p)|^2 d(p) - 2 \sum_{p \sim q} f(p)f(q) = f^T L f = f^T V \Lambda V^T f = \hat{f}^T \hat{\Lambda} \hat{f} = \sum_{\lambda_i \leq \omega} \lambda_i |\hat{f}(i)|^2 \leq \omega \|\hat{f}\|^2.$$  \hspace{1cm} (8)

In the above derivation, $d(p)$ denotes the degree of vertex $p$, and $\hat{f}$ denotes the Fourier transform of $f$. The last inequality is because the components of $\hat{f}$ corresponding to the frequencies higher than $\omega$ are zero for $f \in PW_\omega(G)$.

Combining (5), (7) and (8), we have

$$\|f - Af\|^2 \leq \sum_{u \in \mathcal{S}} \left( K(u) R(u) \sum_{p \sim q} |f(p) - f(q)|^2 \right) \leq Q_{\max} \sum_{p \sim q} |f(p) - f(q)|^2 \leq Q_{\max} \omega \|f\|^2$$

and Lemma 1 is proved.

**B. Proof of Proposition 1**

**Proof:** By the definition of $A$, the iteration can be written as

$$f^{(0)} = Af$$

and

$$f^{(k+1)} = f^{(k)} + A(f - f^{(k)}).$$

Note that $f \in PW_\omega(G)$ and $f^{(k)} \in PW_\omega(G)$ for any $k$, then $f^{(k)} - f \in PW_\omega(G)$. As a consequence of Lemma 1,

$$\|f^{(k+1)} - f\| = \|(f^{(k)} - f) - A(f^{(k)} - f)\| \leq \gamma \|f^{(k)} - f\|,$$

proposition 1 is proved.
C. Proof of Proposition 2

Proof: By the definition of \( A \)

\[
Af = \sum_{u \in S} \langle f, \delta_u \rangle P_\omega(\delta_{N(u)})
\]

\[
= \sum_{u \in S} \langle P_\omega(f), \delta_u \rangle P_\omega(\delta_{N(u)})
\]

\[
= \sum_{u \in S} \langle f, P_\omega(\delta_u) \rangle P_\omega(\delta_{N(u)}).
\]

According to Lemma 1,

\[
\|f - \sum_{u \in S} \langle f, P_\omega(\delta_u) \rangle P_\omega(\delta_{N(u)})\| \leq \gamma \|f\|.
\] (9)

For all \( f \in PW_\omega(\mathcal{G}) \) and \( \{\lambda_u\}_{u \in S} \), we have

\[
\sum_{u \in S} |\langle f, P_\omega(\delta_u) \rangle|^2 = \sum_{u \in S} |f(u)|^2 \leq \|f\|^2.
\] (10)

and

\[
\|\sum_{u \in S} \lambda_u P_\omega(\delta_{N(u)})\|^2 = \|P_\omega \left( \sum_{u \in S} \lambda_u \delta_{N(u)} \right) \|^2
\]

\[
\leq \|\sum_{u \in S} \lambda_u \delta_{N(u)}\|^2 = \sum_{u \in S} |N(u)| \cdot |\lambda_u|^2 \leq N_{\text{max}} \sum_{u \in S} |\lambda_u|^2.
\] (11)

Combining (9), (10) and (11) and Proposition 2 in [6], \( \{P_\omega(\delta_{N(u)})\}_{u \in S} \) is a frame with bounds \((1 - \gamma)^2 \) and \( N_{\text{max}} \), and \( \{P_\omega(\delta_u)\}_{u \in S} \) is a frame with bounds \((1 - \gamma)^2 / N_{\text{max}} \) and 1. Proposition 2 is proved. ■

D. Proof of Lemma 2

Proof: According to Lemma 1 and Proposition 1, we have \( \|I - A\| \leq \gamma < 1 \) when \( \gamma = \sqrt{Q_{\text{max}}\omega} < 1 \).

Then \( A \) is invertible and \( 1 - \gamma \leq \|A\| \leq 1 + \gamma \).

\[
\|f\|^2 = \|A^{-1}Af\|^2 \leq (1 - \gamma)^{-2} \|Af\|^2
\]

\[
\leq (1 - \gamma)^{-2} \|\sum_{u \in S} f(u)\delta_{N(u)}\|^2
\]

\[
= (1 - \gamma)^{-2} \sum_{u \in S} |N(u)| \cdot |f(u)|^2.
\]

Then the left inequality of Lemma 2 is proved.

From the proof of Lemma 1, it is known that

\[
\|f - \sum_{u \in S} f(u)\delta_{N(u)}\|^2 \leq \gamma \|f\|.
\]

July 24, 2014
Therefore,
\[
\sum_{u \in S} |\mathcal{N}(u)| \cdot |f(u)|^2 = \| \sum_{u \in S} f(u) \delta_{\mathcal{N}(u)} \|^2 \\
\leq \left( \| f \| + \| f - \sum_{u \in S} f(u) \delta_{\mathcal{N}(u)} \| \right)^2 \\
\leq (1 + \gamma)^2 \| f \|^2,
\]
which is the right inequality of Lemma 2.

Considering
\[
|\langle f, \sqrt{\mathcal{N}(u)} \mathcal{P}_\omega(\delta_u) \rangle|^2 = |\mathcal{N}(u)| \cdot |\langle \mathcal{P}_\omega(f), \delta_u \rangle|^2 \\
= |\mathcal{N}(u)| \cdot |f(u)|^2,
\]
The inequalities imply that \( \{ \sqrt{\mathcal{N}(u)} \mathcal{P}_\omega(\delta_u) \}_{u \in S} \) is a frame in \( PW_\omega(G) \) with bounds \( (1 - \gamma)^2 \) and \( (1 + \gamma)^2 \).

**E. Proof of Proposition 3**

**Proof:** From Lemma 2, \( \{ \sqrt{\mathcal{N}(u)} \mathcal{P}_\omega(\delta_u) \}_{u \in S} \) is a frame with bounds \( A = (1 - \gamma)^2 \) and \( B = (1 + \gamma)^2 \). By the property of frame [10], the original signal can be reconstructed by
\[
f^{(k+1)} = f^{(k)} + \lambda A_w(f - f^{(k)})
\]
where the frame operator is
\[
A_w f = \sum_{u \in S} \langle f, \sqrt{\mathcal{N}(u)} \mathcal{P}_\omega(\delta_u) \rangle \sqrt{\mathcal{N}(u)} \mathcal{P}_\omega(\delta_u) \\
= \mathcal{P}_\omega \left( \sum_{u \in S} |\mathcal{N}(u)| f(u) \delta_u \right),
\]
and the parameter \( \lambda \) is chosen as
\[
\lambda = \frac{2}{A + B} = \frac{1}{1 + \gamma^2}.
\]
The property of frame [10] shows that the iteration satisfies
\[
\|f^{(k)} - f\| \leq \left( \frac{B - A}{B + A} \right)^k \|f^{(0)} - f\| = \left( \frac{2\gamma}{1 + \gamma^2} \right)^k \|f^{(0)} - f\|.
\]
Then Proposition 3 is proved. \( \blacksquare \)
REFERENCES


