On the Null Space Constant for $\ell_p$ Minimization

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Abstract

The literature on sparse recovery often adopts the $\ell_p$ “norm” ($p \in [0,1]$) as the penalty to induce sparsity of the signal satisfying an underdetermined linear system. The performance of the corresponding $\ell_p$ minimization problem can be characterized by its null space constant. In spite of the NP-hardness of computing the constant, its properties can still help in illustrating the performance of $\ell_p$ minimization. In this letter, we show the strict increase of the null space constant in the sparsity level $k$ and its continuity in the exponent $p$. We also indicate that the constant is strictly increasing in $p$ with probability 1 when the sensing matrix $A$ is randomly generated. Finally, we show how these properties can help in demonstrating the performance of $\ell_p$ minimization, mainly in the relationship between the the exponent $p$ and the sparsity level $k$.

Keywords: Sparse recovery, null space constant, $\ell_p$ minimization, monotonicity, continuity.

1 Introduction

An important problem that often arises in signal processing, machine learning, and statistics is sparse recovery [1–3]. It is in general formulated in the standard form

$$\arg\min_{x} \|x\|_0 \text{ subject to } Ax = y \quad (1)$$

where the sensing matrix $A \in \mathbb{R}^{M \times N}$ has more columns than rows and the $\ell_0$ “norm” $\|x\|_0$ denotes the number of nonzero entries of the vector $x$. The combinatorial optimization (1) is NP-hard and therefore cannot be solved efficiently [4]. A standard method to solve this problem is by relaxing the non-convex discontinuous $\ell_0$ “norm” to the convex $\ell_1$ norm [5], i.e.,

$$\arg\min_{x} \|x\|_1 \text{ subject to } Ax = y. \quad (2)$$

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It is theoretically proved that under some certain conditions \([5,6]\), the optimum solution of (2) is identical to that of (1).

Some works try to bridge the gap between \(\ell_0\) “norm” and \(\ell_1\) norm by non-convex but continuous \(\ell_p\) “norm” \((0 < p < 1)\) \([7–10]\), and consider the \(\ell_p\) minimization problem

\[
\arg\min_x \|x\|_p \quad \text{subject to} \quad Ax = y
\]

where \(\|x\|_p = \sum_{i=1}^{N} |x_i|^p\). Though finding the global optimal solution of \(\ell_p\) minimization is still NP-hard, computing a local minimizer can be done in polynomial time \([11]\). The global optimality of (3) has been studied and various conditions have been derived, for example, those based on restricted isometry property \([7–9,12]\) and null space property \([10,13]\). Among them, a necessary and sufficient condition is based on the null space property and its constant \([10,13,14]\).

**Definition 1.** For any \(0 \leq p \leq 1\), define null space constant \(\gamma(\ell_p, A, k)\) as the smallest quantity such that

\[
\sum_{i \in S} |z_i|^p \leq \gamma(\ell_p, A, k) \sum_{i \not\in S} |z_i|^p
\]

holds for any set \(S \subset \{1, 2, \ldots, N\}\) with \(#S \leq k\) and for any vector \(z \in N(A)\) which denotes the null space of \(A\).

It has been shown that for any \(p \in [0, 1]\), \(\gamma(\ell_p, A, k) < 1\) is a necessary and sufficient condition such that for any \(k\)-sparse \(x^*\) and \(y = Ax^*\), \(x^*\) is the unique solution of \(\ell_p\) minimization \([10]\). Therefore, \(\gamma(\ell_p, A, k)\) is a tight quantity in indicating the performance of \(\ell_p\) minimization \((0 \leq p \leq 1)\) in sparse recovery. However, it has been shown that calculating \(\gamma(\ell_p, A, k)\) is in general NP-hard \([15]\), which makes it difficult to check whether the condition is satisfied or violated. Despite this, properties of \(\gamma(\ell_p, A, k)\) are of tremendous help in illustrating the performance of \(\ell_p\) minimization, e.g., non-decrease of \(\gamma(\ell_p, A, k)\) in \(p \in [0, 1]\) shows that if \(\ell_p\) minimization guarantees successful recovery of all \(k\)-sparse signal and \(0 \leq q \leq p\), then \(\ell_q\) minimization also does \([10]\).

In this letter, we give some new properties of the null space constant \(\gamma(\ell_p, A, k)\). Specifically, we prove that \(\gamma(\ell_p, A, k)\) is strictly increasing in \(k\) and is continuous in \(p\). For random sensing matrix \(A\), the non-decrease of \(\gamma(\ell_p, A, k)\) in \(p\) can be improved to strict increase with probability 1. Based on them, the performance of \(\ell_p\) minimization can be intuitively demonstrated and understood.

## 2 Main Contribution

This section introduces some properties of null space constant \(\gamma(\ell_p, A, k)\) \((0 \leq p \leq 1)\). We begin with a lemma about \(\gamma(\ell_p, A, k)\) which will play a central role in the theoretical analysis. The spark of a matrix \(A\), denoted as \(\text{Spark}(A)\) \([16]\), is the smallest number of columns from \(A\) that are linearly dependent.
**Lemma 1.** Suppose Spark(A) = L + 1. For p ∈ [0, 1],

1) \( \gamma(\ell_p, A, k) \) is finite if and only if \( k \leq L \);

2) For \( k \leq L \), there exist \( S' \subset \{1, 2, \ldots, N\} \) with \( \#S' \leq k \) and \( z' \in N(A) \setminus \{0\} \) such that

\[
\sum_{i \in S'} |z'_i|^p = \gamma(\ell_p, A, k) \sum_{i \notin S'} |z'_i|^p
\]  

(5)

**Proof.** See Section 3.1.

First, we show the strict increase of \( \gamma(\ell_p, A, k) \) in \( k \).

**Theorem 1.** Suppose Spark(A) = L+1. Then for \( p \in [0, 1] \), \( \gamma(\ell_p, A, k) \) is strictly increasing in \( k \) when \( k \leq L \).

**Proof.** See Section 3.2.

**Remark 1.** For any \( p \in [0, 1] \), we can define a set \( K_p(A) \) of all positive integers \( k \) that every \( k \)-sparse \( x^* \) can be recovered as the unique solution of \( \ell_p \) minimization (3) with \( y = Ax^* \). According to Theorem 1, \( K_p(A) \) contains successive integers starting from 1 to some integer \( k^*_p(A) \) and is possibly empty.

**Remark 2.** If Spark(A) = L + 1, then \( k^*_0(A) \) is \( \lfloor L/2 \rfloor \) [16]. Therefore, if \( L \geq 2 \), \( k^*_0(A) \geq 1 \).

**Remark 3.** For \( A \) with identical column norms, if Spark(A) = L + 1 and \( L \geq 2 \), then \( k^*_1(A) \geq 1 \). To show this, we only need to prove that \( \gamma(\ell_1, A, 1) < 1 \). First, for any \( 1 \leq i \leq N \) and \( z \in N(A) \setminus \{0\} \), since \( Az = 0 \), \( z_i a_i = -\sum_{j \neq i} z_j a_j \) where \( a_i \) is the \( i \)th column of \( A \). Since

\[
|z_i| \cdot \|a_i\|_2 = \|z_i a_i\|_2 = \left\| \sum_{j \neq i} z_j a_j \right\|_2 \leq \sum_{j \neq i} |z_j| \cdot \|a_j\|_2
\]

with equality holds only when \( z_j a_j (j \neq i) \) are all on the same ray, which cannot be true since \( \text{Spark}(A) = L + 1 \geq 3 \). Since \( A \) has identical column norms, \( |z_i| < \sum_{j \neq i} |z_j| \) holds, which leads to \( \gamma(\ell_1, A, 1) < 1 \) because of Lemma 1.2).

Now we turn to the properties of \( \gamma(\ell_p, A, k) \) as a function of \( p \). The following result reveals the continuity of \( \gamma(\ell_p, A, k) \) in \( p \).

**Theorem 2.** Suppose Spark(A) = L + 1. Then for \( k \leq L \), \( \gamma(\ell_p, A, k) \) is a continuous function in \( p \in [0, 1] \).

**Proof.** See Section 3.3.

**Remark 4.** Some works have discussed the equivalence of \( \ell_0 \) and \( \ell_p \) minimizations. In [17], it is shown that the sufficient condition for the equivalence of these two minimization problems approaches the necessary and sufficient condition for the uniqueness of solutions of
Figure 1: The figure shows $k_p^*(A)$ as a function of $p$, where the argument $A$ is omitted for concision.

$\ell_0$ minimization. In [7], it is shown that for any $k$-sparse $x^*$ and $y = Ax^*$, if $\delta_{2k+1} < 1$, then there is $p > 0$ such that $x^*$ is the unique solution of $\ell_p$ minimization. This result is improved to $\delta_{2k} < 1$ which is optimal since it is exactly the necessary and sufficient condition for $x^*$ being the unique solution of $\ell_0$ minimization [12]. [18] shows the equivalence of the $\ell_0$- and $\ell_p$-norm minimization problem for sufficiently small $p$. According to Theorem 2, we can also justify this result: For any $k$-sparse $x^*$ and $y = Ax^*$, if $\gamma(\ell_p, A, k) < 1$, then there is $p > 0$ such that $\gamma(\ell_p, A, k) < 1$ and $x^*$ is the unique solution of $\ell_p$ minimization.

**Remark 5.** In [10], the author defines a set $\mathcal{P}_k(A)$ of reconstruction exponents, that is the set of all exponents $0 < p \leq 1$ for which every $k$-sparse $x^*$ is recovered as the unique solution of $\ell_p$ minimization with $y = Ax^*$. It is shown that $\mathcal{P}_k(A)$ is a (possibly empty) open interval $(0, p_k^*(A))$ [10]. This result can be easily shown by Theorem 2. Since $\gamma(\ell_p, A, k)$ is a non-decreasing [13] continuous function in $p \in [0, 1]$, the inverse image of the open interval $(-\infty, 1)$ is also an open interval of $[0, 1]$. Therefore, the requirement that $\gamma(\ell_p, A, k) < 1$ is equivalent to $p \in [0, p_k^*(A))$.

**Remark 6.** For any $A$, we can plot $k_p^*(A)$ as a function of $p$, as shown in Fig. 1. For concision, we omit the argument $A$ in the figure. It is obvious that $k_p^*(A)$ is a step function decreasing from $k_0^*(A)$ to $k_1^*(A)$. Three facts needs to be pointed out. First, $k_p^*(A)$ is right-continuous, which is an easy consequence of Theorem 2. Second, the points $(p_0, k_0)$ corresponding to the hollow circles in Fig. 1 satisfy $\gamma(\ell_{p_0}, A, k_0) = 1$. Third, for the $p$-axis $p_0$ of the points of discontinuity, the one-sided limits satisfy $\lim_{p \rightarrow p_0^-} k_p^*(A) = \lim_{p \rightarrow p_0^+} k_p^*(A) = 1$. This can be proved by Theorem 1 that if $\gamma(\ell_{p_0}, A, k_0) = 1$, then $\gamma(\ell_{p_0}, A, k_0 - 1) < 1$.

Finally, we introduce an important property of $\gamma(\ell_p, A, k)$ as a function of $p$ with regard to random matrix $A$. 

\begin{align*}
\ell_0 \text{ minimization. In [7], it is shown that for any } k \text{-sparse } x^* \text{ and } y = Ax^*, \text{ if } \delta_{2k+1} < 1, \text{ then there is } p > 0 \text{ such that } x^* \text{ is the unique solution of } \ell_p \text{ minimization. This result is improved to } \delta_{2k} < 1 \text{ which is optimal since it is exactly the necessary and sufficient condition for } x^* \text{ being the unique solution of } \ell_0 \text{ minimization [12]. [18] shows the equivalence of the } \ell_0- \text{ and } \ell_p\text{-norm minimization problem for sufficiently small } p. \text{ According to Theorem 2, we can also justify this result: For any } k \text{-sparse } x^* \text{ and } y = Ax^*, \text{ if } \gamma(\ell_p, A, k) < 1, \text{ then there is } p > 0 \text{ such that } \gamma(\ell_p, A, k) < 1 \text{ and } x^* \text{ is the unique solution of } \ell_p \text{ minimization.}
\end{align*}
Theorem 3. Suppose the entries of $A \in \mathbb{R}^{M \times N}$ are i.i.d. and satisfy a continuous probability distribution. Then for $k \leq M$, $\gamma(\ell_p, A, k)$ is strictly increasing in $p \in [0,1]$ with probability one.

Proof. See Section 3.4. \hfill \Box

Remark 7. It needs to be noted that there exists $A$ such that $\gamma(\ell_p, A, k)$ is a constant number for all $p \in [0,1]$. For example, for

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

Spark$(A) = 2$. Since $\mathcal{N}(A) = \text{span}([1, -1]^T)$, it is easy to check that for all $p \in [0,1]$, $\gamma(\ell_p, A, 1) = 1$.

Remark 8. To sum up, we can schematically show $\gamma(\ell_p, A, k)$ as a function of $p$ for different $k$ in Fig. 2. According to Theorem 1, these curves are strictly in order without intersections. Theorem 2 reveals that $\gamma(\ell_p, A, k)$ is continuous in $p$. For a random matrix $A$ with i.i.d. entries satisfying a continuous probability distribution, $\gamma(\ell_p, A, k)$ is strictly increasing in $p$ with probability 1 by Theorem 3. According to the definition of $k^*_k(A)$, the curves intersecting $\gamma(\ell_p, A, k) = 1$ ($0 \leq p \leq 1$) are those with $k^*_k(A) + 1 \leq k \leq k^*_0(A)$. According to the definition of $p^*_k(A)$, the $p$-axis of these intersections are $p^*_{k^*_0-1}$, $p^*_{k^*_0}$, $p^*_{k^*_1+1}$ from left to right. Therefore, it is easy to derive Fig. 1 based on Fig. 2 when $A$ is a random matrix.

Figure 2: This figure shows a diagrammatic sketch of $\gamma(\ell_p, A, k)$ as a function of $p$ for different $k$ when $A$ is a random matrix.
3 Proofs

3.1 Proof of Lemma 1

Proof. 1) Since Spark(\(A\)) = \(L + 1\), \(N(\mathbf{A})\) contains an \((L + 1)\)-sparse signal, and it is easy to show that for any \(k \geq L + 1\), \(\gamma(\ell_p, A, k) = +\infty\) according to Definition 1. Next we prove that for \(k \leq L\), \(\gamma(\ell_p, A, k)\) is finite. Define

\[
\theta(p, \mathbf{z}, S) = \frac{\sum_{i \in S} |z_i|^p}{\sum_{i \in S} |z_i|^p}
\]

and \(\mathcal{N}_1(\mathbf{A}) = N(\mathbf{A}) \cap \{\mathbf{z} : \|\mathbf{z}\|_2 = 1\}\) which is a compact set. Then it is easy to see that the definition of null space constant is equivalent to

\[
\gamma(\ell_p, A, k) = \max_{\#S \leq k} \sup_{\mathbf{z} \in \mathcal{N}_1(\mathbf{A})} \theta(p, \mathbf{z}, S).
\]

If \(\gamma(\ell_p, A, k)\) is not finite, then there exists \(S'\) with \(\#S' \leq k\) such that \(\sup_{z \in \mathcal{N}_1(\mathbf{A})} \theta(p, \mathbf{z}, S')\) is not finite. Therefore, for any \(n \in \mathbb{N}^+\), there exists \(\mathbf{z}^{(n)} \in \mathcal{N}_1(\mathbf{A})\) such that

\[
\theta(p, \mathbf{z}^{(n)}, S') \geq n.
\]

If \(p = 0\), since \(\mathbf{z}^{(n)}\) is at least \((L + 1)\)-sparse, it is easy to see that \(\theta(0, \mathbf{z}^{(n)}, S') \leq k\) holds for any \(n \in \mathbb{N}^+\). This contradicts (9) when \(n > k\). If \(p \in (0, 1]\), according to Lemma 4.5 in [10], \(\|\mathbf{z}^{(n)}\|_p \leq N^{\frac{1}{p} - \frac{1}{2}} \|\mathbf{z}^{(n)}\|_2 = N^{\frac{1}{p} - \frac{1}{2}}\), and (9) implies

\[
\sum_{i \in S'} |z_i^{(n)}|^p \leq \frac{N^{1 - \frac{p}{2}}}{n + 1}.
\]

Due to the compactness of \(\mathcal{N}_1(\mathbf{A})\), the sequence \(\{\mathbf{z}^{(n)}\}_n\) has a convergent subsequence \(\{\mathbf{z}^{(m)}\}_m\), and its limit \(\mathbf{z}'\) also lies in \(\mathcal{N}_1(\mathbf{A})\). Then (10) implies \(z_i' = 0\) for \(i \not\in S'\), i.e., \(\mathcal{N}_1(\mathbf{A})\) contains a \(k\)-sparse element \(\mathbf{z}'\). This contradicts the assumption that \(\text{Spark}(\mathbf{A}) = L + 1 > k\).

2) If \(p = 0\), for any \(S\) with \(\#S \leq k\) and any \(\mathbf{z} \in \mathcal{N}(\mathbf{A}) \setminus \{\mathbf{0}\}\), it holds that

\[
\theta(0, \mathbf{z}, S) \leq \frac{k}{L + 1 - k}.
\]

On the other hand, since \(\text{Spark}(\mathbf{A}) = L + 1\), \(\mathcal{N}(\mathbf{A})\) contains an \((L + 1)\)-sparse signal \(\mathbf{z}'\) with \(T\) as its support set. For any \(S' \subseteq T\) with \(\#S' = k\), \(\theta(0, \mathbf{z}', S') = k/(L + 1 - k)\), and therefore (5) holds.

If \(p \in (0, 1]\), recalling the equivalent definition (8), there exists \(S'\) with \(\#S' \leq k\) such that

\[
\gamma(\ell_p, A, k) = \sup_{\mathbf{z} \in \mathcal{N}_1(\mathbf{A})} \theta(p, \mathbf{z}, S').
\]

Since \(\mathcal{N}_1(\mathbf{A})\) is compact and the function \(\theta(p, \mathbf{z}, S')\) is continuous in \(\mathbf{z}\) on \(\mathcal{N}_1(\mathbf{A})\), it is easy to show that there exists \(\mathbf{z}' \in \mathcal{N}_1(\mathbf{A})\) such that \(\gamma(\ell_p, A, k) = \theta(p, \mathbf{z}', S')\). \(\square\)
3.2 Proof of Theorem 1

Proof. We prove that when \( p \in [0, 1] \) and \( 2 \leq k \leq L \),

\[
\gamma(\ell_p, \mathbf{A}, k - 1) < \gamma(\ell_p, \mathbf{A}, k). \tag{13}
\]

According to Lemma 1.2), there exist \( S' \) with \( \#S' \leq k - 1 \) and \( \mathbf{z}' \in \mathcal{N}_1(\mathbf{A}) \) such that

\[
\gamma(\ell_p, \mathbf{A}, k - 1) = \theta(p, \mathbf{z}', S'). \tag{14}
\]

Since \( \mathbf{z}' \) is at least \((L + 1)\)-sparse, there exists an index \( s' \in \{1, 2, \ldots, N\} \setminus S' \) such that \( z_{s'}' \neq 0 \). Let \( S'' = S' \cup \{s'\} \), then

\[
\sum_{i \in S'} |z_i'|^p < \sum_{i \in S''} |z_i'|^p, \quad \sum_{i \notin S'} |z_i'|^p > \sum_{i \notin S''} |z_i'|^p > 0 \tag{15}
\]

and hence

\[
\theta(p, \mathbf{z}', S') < \theta(p, \mathbf{z}', S''). \tag{16}
\]

Recalling (14) and the equivalent definition (8), we can get (13) and complete the proof. \( \square \)

3.3 Proof of Theorem 2

Proof. According to Theorem 5 in [13], \( \gamma(\ell_p, \mathbf{A}, k) \) is non-decreasing in \( p \in [0, 1] \) and therefore can only have jump discontinuities. We show this is impossible by two steps.

First, for any \( p \in (0, 1] \), we prove the one-sided limit from the negative direction satisfies

\[
L^- := \lim_{q \to p^-} \gamma(\ell_q, \mathbf{A}, k) = \gamma(\ell_p, \mathbf{A}, k). \tag{17}
\]

According to Lemma 1.2), there exist \( S' \) with \( \#S' \leq k \) and \( \mathbf{z}' \in \mathcal{N}_1(\mathbf{A}) \) satisfying

\[
\gamma(\ell_p, \mathbf{A}, k) = \theta(p, \mathbf{z}', S'). \tag{18}
\]

According to the definition of \( \theta(p, \mathbf{z}, S) \), it is easy to show that

\[
\lim_{q \to p^-} \theta(q, \mathbf{z}', S') = \theta(p, \mathbf{z}', S'), \tag{19}
\]

and then (17) holds obviously.

Second, for any \( p \in [0, 1) \), we prove the one-sided limit from the positive direction satisfies

\[
L^+ := \lim_{q \to p^+} \gamma(\ell_q, \mathbf{A}, k) = \gamma(\ell_p, \mathbf{A}, k). \tag{20}
\]

Since \( p < 1 \), there exists \( N_0 \in \mathbb{N}^+ \) such that \( p + N_0^{-1} \leq 1 \). Then for \( n \geq N_0 \), Lemma 1.2) reveals that there exist \( S^{(n)} \) with \( \#S^{(n)} \leq k \) and \( \mathbf{z}^{(n)} \in \mathcal{N}_1(\mathbf{A}) \) such that

\[
\gamma(\ell_{p+n^{-1}}, \mathbf{A}, k) = \theta(p + n^{-1}, \mathbf{z}^{(n)}, S^{(n)}). \tag{21}
\]
Since there are only finite different $S$ satisfying $\#S \leq k$, there exists $S'$ with $\#S' \leq k$ such that an infinite subsequence of $\{z^{(n)}\}_n$ is associated with $S'$. Due to the compactness of $\mathcal{N}_1(\mathbf{A})$, this subsequence has a convergent subsequence $\{z^{(m)}\}_m$, and its limit $z'$ also lies in $\mathcal{N}_1(\mathbf{A})$. According to the definition of $\theta(p, z, S)$ and (21),

$$\theta(p, z', S') = \lim_{m \to +\infty} \theta(p + n_m^{-1}, z^{(m)}), S') = L^+, \quad (22)$$

and consequently $\gamma(\ell_p, \mathbf{A}, k) \geq L^+$. Since $\gamma(\ell_p, \mathbf{A}, k)$ is non-decreasing in $p$, $\gamma(\ell_p, \mathbf{A}, k) \leq L^+$ and (20) is proved. \hfill \Box

### 3.4 Proof of Theorem 3

**Proof.** First, we show that $\text{Spark}(\mathbf{A}) = M+1$ with probability 1. Let $\mathcal{M}(M)$ denote the $M^2$-dimensional vector space of $M \times M$ real matrices. For any $0 \leq k \leq M$, let $\mathcal{M}_k(M)$ denote the subset of $\mathcal{M}(M)$ consisting of matrices of rank $k$. It can be proved that $\mathcal{M}_k(M)$ is an embedded submanifold of dimension $k(2M - k)$ in $\mathcal{M}(M)$ [19]. Consequently, for $M \times M$ matrices with i.i.d. entries drawn from a continuous distribution, the $M^2$-dimensional volume of the set of singular matrices $\bigcup_{k=0}^{M-1} \mathcal{M}_k(M)$ is zero. In other words, any $M$, or fewer, random vectors in $\mathbb{R}^M$ with i.i.d. entries drawn from a continuous distribution are linearly independent with probability 1. On the other hand, more than $M$ vectors in $\mathbb{R}^M$ are always linearly dependent. Therefore, $\text{Spark}(\mathbf{A}) = M+1$ with probability 1.

Next, with the equivalent definition (8), we prove that for $k \leq M$ and $0 \leq p < q \leq 1$,

$$\max_{\#S \leq k} \sup_{z \in \mathcal{N}_1(\mathbf{A})} \theta(p, z, S) < \max_{\#S \leq k} \sup_{z \in \mathcal{N}_1(\mathbf{A})} \theta(q, z, S) \quad (23)$$

holds with probability 1. According to Lemma 1.2, there exist $S'$ with $\#S' \leq k$ and $z' \in \mathcal{N}_1(\mathbf{A})$ such that

$$\theta(p, z', S') = \max_{\#S \leq k} \sup_{z \in \mathcal{N}_1(\mathbf{A})} \theta(p, z, S). \quad (24)$$

Suppose $z'$ has $N_s$ nonzero entries with $T$ as its support set, then $N_s \geq M+1$ with probability 1. It is obvious that $S' \subset T$, and for any $i \in S'$ and any $l \in T \setminus S'$, $|z'_i| \geq |z'_l| > 0$. Since $p < q$, $|z'_i|^{q-p} \geq |z'_l|^{q-p}$ and therefore

$$|z'_i|^q |z'_l|^p \geq |z'_l|^q |z'_l|^p. \quad (25)$$

Summing (25) with $i$ in $S'$ and $l$ in $T \setminus S'$, we can obtain

$$\sum_{i \in S'} |z'_i|^q \sum_{l \in T \setminus S'} |z'_l|^p \geq \sum_{i \in S'} |z'_i|^p \sum_{l \in T \setminus S'} |z'_l|^q \quad (26)$$

which is equivalent to

$$\theta(p, z', S') \leq \theta(q, z', S'). \quad (27)$$
Since $p < q$, it is easy to check that the equality in (27) holds only when $|z_i'| = |z_l'|$ for all $i \in S'$ and all $l \in T \setminus S'$, i.e., the nonzero entries of $z'$ have the same magnitude. We prove that $N_1(A)$ contains such $z'$ with probability 0, which together with (24) imply that

$$\gamma(\ell_p, A, k) = \theta(p, z', S') < \theta(q, z', S') \leq \gamma(\ell_q, A, k)$$

(28)

holds with probability 1.

To this end, let $\mathcal{M}(M, N)$ denote the $MN$-dimensional vector space of $M \times N$ real matrices. For fixed $z \in \mathbb{R}^N$ with $\|z\|_2 = 1$, it can be easily shown that the subset

$$\mathcal{M}_z(M, N) = \{ A \in \mathcal{M}(M, N) : Az = 0 \}$$

(29)

is an $M(N-1)$-dimensional subspace in $\mathcal{M}(M, N)$. Therefore, for $A \in \mathcal{M}(M, N)$ with i.i.d. entries drawn from a continuous probability distribution, $N_1(A)$ contains $z$ with probability 0. In $\{ z \in \mathbb{R}^N : \|z\|_2 = 1 \}$, the number of vectors whose nonzero entries have the same magnitude is

$$\sum_{i=1}^{N} \binom{N}{i} 2^i = 3^N - 1$$

(30)

which is a finite number. Therefore, with probability 0, $N_1(A)$ contains a vector $z'$ which makes the equality in (27) hold. That is, $\gamma(\ell_p, A, k)$ is strictly increasing in $p \in [0, 1]$ with probability 1.

\[ \Box \]

4 Conclusion

In characterizing the performance of $\ell_p$ minimization in sparse recovery, null space constant $\gamma(\ell_p, A, k)$ can be served as a necessary and sufficient condition for the perfect recovery of all $k$-sparse signals. This letter derives some basic properties of $\gamma(\ell_p, A, k)$ in $k$ and $p$. In particular, we show that $\gamma(\ell_p, A, k)$ is strictly increasing in $k$ and is continuous in $p$, meanwhile for random $A$, the constant is strictly increasing in $p$ with probability 1. Possible future works include the properties of $\gamma(\ell_p, A, k)$ in $A$, for example, the requirement of number of measurements $M$ to guarantee $\gamma(\ell_p, A, k) < 1$ with high probability when $A$ is randomly generated.

References


