

Retrieval of Sparse Solutions of Multiple-Measurement Vectors via Zero-point Attracting Projection

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Abstract

A new sparse signal recovery algorithm for multiple-measurement vectors (MMV) problem is proposed in this paper. The sparse representation is iteratively drawn based on the idea of zero-point attracting projection (ZAP). In each iteration, the solution is first updated along the negative gradient direction of an approximate $\ell_{2,0}$ norm to encourage sparsity, and then projected to the solution space to satisfy the under-determined equation. A variable step size scheme is adopted further to accelerate the convergence as well as to improve the recovery accuracy. Numerical simulations demonstrate that the performance of the proposed algorithm exceeds the references in various aspects, as well as when applied to the Modulated Wideband Converter, where recovering MMV problem is crucial to its performance.

Keywords: compressed sensing, sparse recovery, multiple-measurement vectors, zero-point attracting projection, the Modulated Wideband Converter.

1 Introduction

Sparse recovery is one of the essential issues in many fields of signal processing, including compressed sampling (CS) [1, 2], which is a novel sampling theory. In some applications such as magnetoencephalography (MEG) [3], source localization [4] and analog-to-digital conversion [5], the sparse unknowns can be recovered jointly, which results in the problem of multiple-measurement vectors (MMV). In the Modulated Wideband Converter (MWC) [5], the locations of narrow-band signals in a wide spectrum range are detected by solving the MMV problem, which is of great significance to the performance of the conversion system.

Several algorithms have been proposed to derive the sparse solution to the MMV problem. The OMP algorithm for MMV [6] finds a sparse solution by sequentially building up a small subset of column vectors selected from \mathbf{A} to represent \mathbf{Y} . FOCal Underdetermined

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System Solver (FOCUSS) methods [3] compute the constraint solution in an iterative procedure using the standard method of Lagrange multipliers. ReMBo [7] reduces MMV to SMV by multiplying a random column vector drawn from an absolutely continuous distribution. Recently, a robust algorithm for joint-sparse recovery named Joint $\ell_{2,0}$ Approximation algorithm (JLZA) [8] has been proposed to calculate the solution in a fixed point iteration with an approximation of $\ell_{2,0}$ norm. Different algorithms which aim at solving the problem of MMV are surveyed and compared in [9].

This work extends a recently proposed zero-point attracting projection (ZAP) algorithm [10] to the MMV scenario. Derived from the adaptive filtering framework, ZAP defines the cost function as the sparsity and updates the sparse representation in the solution space. Specifically, starting from the least squares solution, it is modified in a fixed step-size along the negative gradient direction of an approximate ℓ_0 norm towards the zero point, and then projected back to the solution space again. This operation is executed iteratively until the stopping condition is satisfied. In this paper, an approximate $\ell_{2,0}$ norm is adopted in ZAP and the recursive solution to MMV is derived. In addition, the step-size of zero-point attraction is varied dynamically to accelerate the convergence. The rest of the present paper is organized as follows. Section 2 formulates the MMV problem and the approximate $\ell_{2,0}$ norm. Section 3 introduces the ZAP algorithm for MMV, and further discussions and improvements are given in Section 4. Section 5 shows the simulation results and this paper is concluded in Section 6.

2 Problem Formulation

The problem of MMV is formulated as $\mathbf{Y} = \mathbf{A}\mathbf{X}$, where $\mathbf{Y} \in \mathbb{R}^{M \times L}$ is the matrix of measurements, $\mathbf{A} \in \mathbb{R}^{M \times N}$ is the sensing matrix, and $\mathbf{X} \in \mathbb{R}^{N \times L}$ is the unknown signal assumed jointly K -sparse, i.e. it has K nonzero rows at most. It is shown [6] that the following problem

$$\hat{\mathbf{X}} = \arg \min_{\mathbf{X}} \|\mathbf{r}(\mathbf{X})\|_0, \text{ subject to } \mathbf{Y} = \mathbf{A}\mathbf{X} \quad (1)$$

finds the unique K -sparse solution provided that

$$K < \frac{\text{Spark}(\mathbf{A}) + \text{Rank}(\mathbf{Y}) - 1}{2},$$

where $\mathbf{r}(\mathbf{X})$ returns a column vector whose i th item is any vector norm of the i th row of \mathbf{X} , $\|\cdot\|_0$ counts nonzero elements of the vector, $\text{Spark}(\mathbf{A})$ is the smallest possible integer such that there exist $\text{Spark}(\mathbf{A})$ columns of matrix \mathbf{A} that are linearly dependent, and $\text{Rank}(\mathbf{Y})$ is the rank of matrix \mathbf{Y} .

According to the references [3, 8], we choose $\ell_{2,0}$ norm

$$J(\mathbf{X}) = \|\mathbf{r}_2(\mathbf{X})\|_0 \quad (2)$$

to describe the joint sparsity of \mathbf{X} , where $\mathbf{r}_2(\mathbf{X})$ denotes a column vector with ℓ_2 norm of the i th row of \mathbf{X} as its i th item. Following [11, 10], the sparsifying penalty can be further approximated by a continuous function

$$J(\mathbf{X}) = \sum_{i=1}^N F_\alpha(\|\mathbf{x}_i\|_2), \quad (3)$$

where \mathbf{x}_i denotes the i th row of \mathbf{X} , and the function $F_\alpha(\cdot)$ is defined as

$$F_\alpha(w) = \begin{cases} 2\alpha|w| - \alpha^2 w^2 & |w| \leq \frac{1}{\alpha}; \\ 1 & \text{elsewhere.} \end{cases} \quad (4)$$

It is readily recognized that $F_\alpha(w)$ converges to the ℓ_0 quasi-norm as $\alpha > 0$ approaches infinity. Since it is non-differentiable at the origin, the derivative of $F_\alpha(\cdot)$ can be replaced by one of its sub-derivatives

$$f_\alpha(w) = \begin{cases} 2\alpha \cdot \text{sgn}(w) - 2\alpha^2 w & |w| \leq \frac{1}{\alpha}; \\ 0 & \text{elsewhere,} \end{cases} \quad (5)$$

where $\text{sgn}(\cdot)$ denotes the sign function with $\text{sgn}(0) = 0$.

3 Zero-point Attracting Projection for MMV

In this section we extend ZAP to the MMV scenario. Further discussions and improvements of this algorithm will be shown in section 4.

ZAP for MMV derives the sparse solution in an iterative procedure. In the n th iteration, the solution is first updated along the negative gradient direction of the sparsifying penalty (3),

$$\tilde{\mathbf{X}}(n) = \mathbf{X}(n-1) - \kappa \cdot \nabla J(\mathbf{X}(n-1)), \quad (6)$$

where $\tilde{\mathbf{X}}(n)$ denotes the temporary solution, $\kappa > 0$ denotes the step size, and $\nabla J(\cdot)$ is a gradient matrix with partial derivative

$$(\nabla J(\mathbf{X}))_{i,j} = \frac{\partial J(\mathbf{X})}{\partial x_{i,j}} \quad (7)$$

as its (i, j) th element, where $x_{i,j}$ stands for the (i, j) th entry of \mathbf{X} . Substituting (4) into (3), and together with (5), the (i, j) th element of the gradient matrix can be calculated

$$(\nabla J(\mathbf{X}))_{i,j} = \frac{\partial F_\alpha(\|\mathbf{x}_i\|_2)}{\partial x_{i,j}} = \begin{cases} \frac{f_\alpha(\|\mathbf{x}_i\|_2)}{\|\mathbf{x}_i\|_2} \cdot x_{i,j} & \|\mathbf{x}_i\|_2 \neq 0; \\ 0 & \|\mathbf{x}_i\|_2 = 0. \end{cases} \quad (8)$$

Then the solution is projected to the solution space to satisfy $\mathbf{Y} = \mathbf{A}\mathbf{X}$

$$\mathbf{X}(n) = \tilde{\mathbf{X}}(n) + \mathbf{A}^\dagger(\mathbf{Y} - \mathbf{A}\tilde{\mathbf{X}}(n)), \quad (9)$$

where $\mathbf{A}^\dagger = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}$ denotes the pseudo-inverse matrix of \mathbf{A} .

Now we have introduced the basic procedure of ZAP for MMV. Compared with solving the MMV problem column by column independently, ZAP for MMV utilizes the information of joint sparsity by means of (8). The updating step size of each element is renormalized according to the norm of the corresponding row, which greatly improves the recovery performance. Detailed discussions about the effect of the parameters and a fast convergence version of the proposed algorithm are given in the following section.

4 Discussions and Improvements

So far, the parameters involved in the ZAP for MMV are the ℓ_0 norm approximation factor α and the step size κ . These parameters should be selected carefully.

The choice of α : The value of α has a great impact on the performance of the proposed algorithm. By (5) we know that large α pulls small values to zero quickly but the range of its influence is small, while a small one has the opposite effect. Furthermore, according to (4), larger α leads to a better approximation of the ℓ_0 norm, but simultaneously produces more local minima in the cost function [8]. In practice, we suggest $1/\alpha$ to be a quarter to a half of the smallest nonzero item of $\mathbf{r}_2(\mathbf{X})$.

The choice of κ : As the step size in the gradient descent iterations, the parameter κ suggests a tradeoff between the speed of convergence and the accuracy of the solution. Big κ indicates faster convergence but less accuracy, while small κ yields more accurate solution but costs more iterations. In order to improve the performance of the proposed algorithm, the idea of variable step size is taken into consideration. The control scheme is rather direct: κ is initialized to be a large value, and reduced by a factor η when the algorithm is convergent. This reduction is conducted until κ is sufficiently small, i.e. $\kappa < \kappa_{\min}$, which means that the recovery error reaches a low level.

The criterion of convergence: It is significantly important to select an appropriate criterion of convergence to make full use of variable step size. Considering the iteration is performed along the negative gradient direction of the sparsifying penalty (3), it is obvious that the algorithm reaches steady-state when the penalty starts increasing. To reduce the complexity of evaluation and improve the stability, the criterion is checked every Q iterations.

The initial value: Following the ZAP algorithm for SMV, the least squares solution is selected as the initial value, i.e. $\mathbf{X}(0) = \mathbf{A}^\dagger \mathbf{Y}$.

The stopping condition: The proposed algorithm stops (a) when κ gets below κ_{\min} or (b) when the number of iterations reaches the bound T .

Finally, the proposed algorithm denoted by ZAPMMV is described in Table 1.

Table 1: The ZAPMMV Algorithm

Initialization

1. Calculate $\mathbf{A}^\dagger = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}$;
2. Set $\mathbf{X}(0) = \mathbf{A}^\dagger\mathbf{Y}$;
3. Set $\alpha, \kappa, \eta, \kappa_{\min}, Q, T$;
4. $n = 1$;

repeat

5. Calculate $\nabla J(\mathbf{X}(n-1))$ by (5) and (8);
6. Update $\mathbf{X}(n)$ by (6);
7. Project $\mathbf{X}(n)$ to the solution space by (9);
8. If $\text{mod}(n, Q) = 0$ and $J(\mathbf{X}(n)) \geq J(\mathbf{X}(n-Q))$
 $\kappa = \eta\kappa$;
- End if
9. Iteration number increases by one: $n = n + 1$;

until $\kappa < \kappa_{\min}$ or $n = T$.

5 Simulations

In this section four experiments are designed to demonstrate the performance of the proposed algorithm in various aspects, including recovery probability versus sparsity and robustness to measurement noise, the running time of the algorithms, as well as recovery performance in the Modulated Wideband Converter [5].

In the first two simulations, five existing algorithms, including OMPMMV [6], ReMBo [7], JLZA [8], $\ell_{2,1}$ Minimization [12], and Reweighted $\ell_{2,1}$ Minimization [13], are simulated in the same scenarios for references. Gaussian sensing matrices are adopted in all experiments. The locations of nonzero rows in the jointly sparse matrix are uniformly chosen at random among all possible choices, and the values of nonzero entries are *i.i.d.* standard normal distribution. We define exact recovery when the relative error $\|\mathbf{X} - \hat{\mathbf{X}}\|_F / \|\mathbf{X}\|_F$ is smaller than 1×10^{-3} , where $\|\cdot\|_F$ denotes the Frobenius norm. The experiments are conducted for 1000 trials to calculate the probability of exact recovery or the average relative error.

In ZAPMMV, we set $\alpha = 1$, $\kappa = 0.1$, $\eta = 0.1$, $Q = 11$, $\kappa_{\min} = 1 \times 10^{-6}$, and $T = 500$. In the reference algorithms, the parameters are selected as suggested to yield the best performance. In ReMBo algorithm, OMP is utilized for sparse recovery, and the maximum number of iterations is 20. In Reweighted $\ell_{2,1}$ Minimization, the reweighting procedure is conducted for 4 iterations.

The first experiment tests the recovery probability with respect to the sparsity. We set $N = 200$, $M = 50$, $L = 10$, and the sparsity K varies from 2 to 50. Figure 1 shows the result in this scenario. As can be seen, the recovery performance of ZAPMMV exceeds

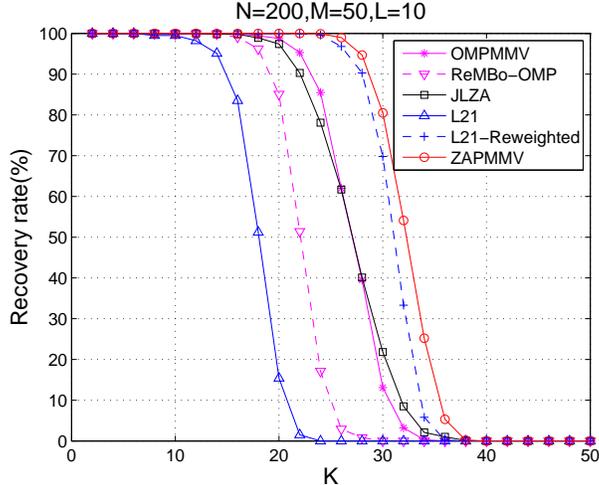


Figure 1: The probability of exact recovery versus sparsity K .

Table 2: The Comparison of Average Running Time (in seconds)

(N, M, K, L)	OMPMMV	JLZA	$\ell_{2,1}$ -Reweighted	ZAPMMV
(1000, 250, 50, 10)	0.22	8.00	9.44	8.04
(2000, 500, 100, 10)	2.03	50.42	65.39	19.03
(3000, 750, 150, 10)	7.96	169.03	213.33	42.19
(4000, 1000, 200, 10)	20.04	433.57	457.91	72.94
(5000, 1250, 250, 10)	40.38	729.52	591.27	106.66

those of the other algorithms.

The second experiment studies the recovery performance against measurement noise. The measurements are contaminated by noise as $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{V}$, where \mathbf{V} denotes the zero mean additive white Gaussian noise. Figure 2 demonstrates the performance of different algorithms in noisy environment, where the measurement signal-to-noise ratio (SNR) varies from 10dB to 50dB. As can be seen from the figure, ZAPMMV is competitive in the presence of noise, and only JLZA outperforms ZAPMMV.

The third experiment briefly compares the running time of the algorithms. Regarding the recovery ability, only OMPMMV, JLZA, Reweighted $\ell_{2,1}$ Minimization ($\ell_{2,1}$ -Reweighted), and ZAPMMV are compared. The simulation is performed for 10 trials without perturbation of noise, and all these algorithms can successfully recover the sparse unknowns. The average running time is shown in Table 2. As can be seen, OMPMMV enjoys the lowest complexity, and ZAPMMV has lower running time than JLZA and Reweighted $\ell_{2,1}$ Minimization as the scale of the MMV problem grows.

In the final experiment, OMPMMV and ZAPMMV are applied to solve the MMV

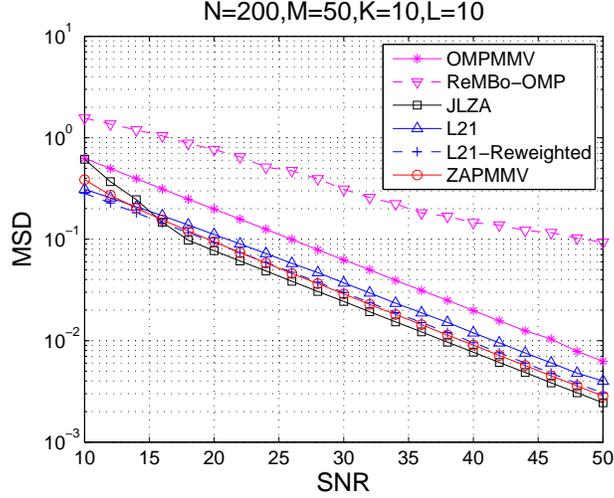


Figure 2: The recovery mean squared deviation (MSD) versus SNR.

problem in the MWC to compare their recovery performance. JLZA is not included since its performance is not so fine in this experiment. In our simulation, 30 channels with 1MHz ADCs are utilized to sample a sparse signal at Nyquist rate 1GHz, and each sampling sequence contains 999 pulses in one period. The original sparse signal contaminated by Gaussian white noise contains 3 narrow-band components (each of 0.5MHz, and unknown location). The waveform and the power spectrum density (PSD) of the input signals with in-band SNR 11dB, 14dB, and 17dB are depicted in Figure 3. Figure 4 demonstrates the probability of successfully detecting the locations of all narrow-band components, two narrow-band components, and only one component, where the average in-band SNR of the original signal varies from 0dB to 23dB. The experiment is conducted for 1000 trials. In this experiment, ZAPMMV recovers the signals with higher probability than the reference method, as verified by its performance against signal noise.

6 Conclusion

In this paper, an approximate $\ell_{2,0}$ norm is adopted to extend the zero-point attracting projection algorithm to solve the multiple-measurement vectors problem. Computer simulations demonstrate that the proposed algorithm outperforms several reference algorithms in respect of sparsity, robustness to measurement noise, running time, and in the application of the Modulated Wideband Converter. Future work may include the theoretic analysis on the convergence and recovery accuracy of the proposed algorithm with respect to initial value and parameters, as well as designing a “noise-aware” version of ZAPMMV through projecting onto the relaxed set $\{\|\mathbf{Y} - \mathbf{A}\mathbf{X}\|_2 \leq \varepsilon\}$.

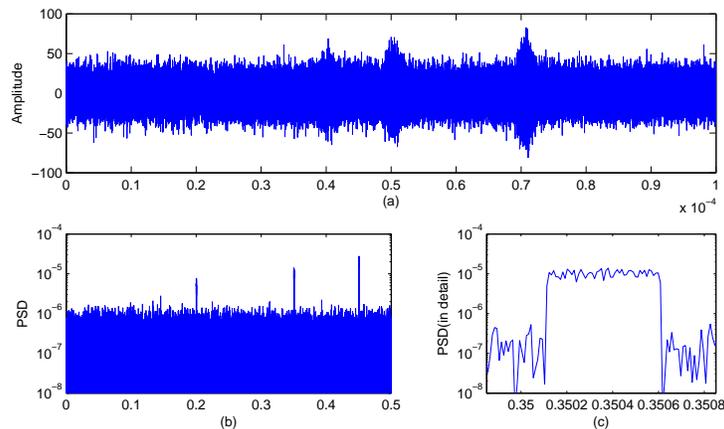


Figure 3: (a) waveform and (b, c) power spectrum density (PSD) of the original signal with in-band SNR 11dB, 14dB, and 17dB, where (c) is the detail of (b). The x-axis of (a) and (b, c) denotes time in seconds and frequency in GHz, respectively.

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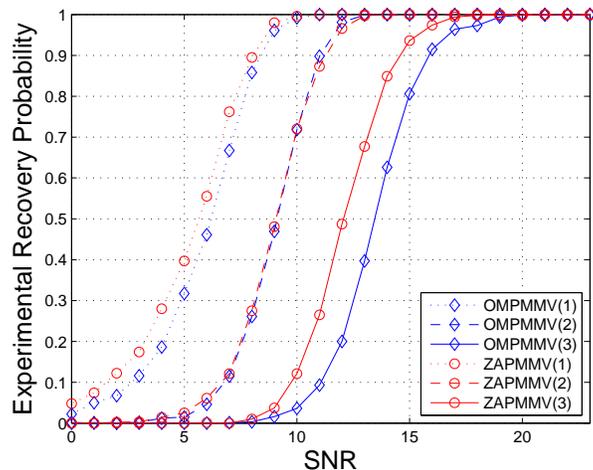


Figure 4: The recovery probability of OMPMMV and ZAPMMV in the MWC simulation, where OMPMMV(i) and ZAPMMV(i) denote that i components are successfully detected.

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