Performance Estimation of Sparse Signal Recovery Under Bernoulli Random Projection with Oracle Information

Ruiyang Song, Laming Chen, and Yuantao Gu

gyt@tsinghua.edu.cn

Department of Electronic Engineering

Tsinghua University, Beijing 100084, CHINA

Beijing, October 18, 2016

Outline

Outline

Robust sparse signal recovery

reconstruct a sparse signal x from measurement y

- $y = Ax + n$ [Candès, Romberg, Tao, 2006; Donoho, 2006]
  - Cramer-Rao bound [Haim, Eldar, 2010]
  - performance of oracle receiver [Coluccia, Aline, Magli, 2014]
  - practical algorithms with reconstruction error bounded by $\|n\|_2$
- $y = A(x + e) + n$ [Davenport, Laska, Treichler, Baraniuk, 2012]
  - noise folding effect [Arias-Castro, Eldar, 2011]
  - equivalent additive noise Ae still independent of x
- $y = (A + E)x + n$ [Ding, Chen, Gu, 2012]
  - equivalent additive noise $Ex + n$ related to x
  - constrained Cramer-Rao bound(CCRB) and Hammersley Chapman-Robbins bound (HCRB) [Tang, Chen, Gu, 2013]

Problem setup

- signal $x = \Psi\theta$ has sparse representation
  - orthonormal $\Psi \in \mathbb{R}^{N\times N}$
  - $\mathrm{supp}(\theta) = \Omega$ and $|\Omega| = K \ll N$
- measurement vector $y = (A + E)x + n$
  - sensing matrix $A \in \mathbb{R}^{M\times N}(K < M < N)$ composed of i.i.d. Bernoulli entries with $p = 1/2$ and sample space $\{-\sigma_A, \sigma_A\}$
  - perturbation matrix $E \in \mathbb{R}^{M\times N}$ with entries i.i.d. $\mathcal{N}(0, \sigma_E^2)$
  - measurement noise $n \sim \mathcal{N}(0, \Sigma_n)$, where $\Sigma_n = \sigma_n^2 \mathbf{I}$
- normalized mean square error (mse_0)

$$\mathrm{mse}_0 = \mathbb{E}\|x - \widehat{x}\|_2^2 / \mathbb{E}\|x\|_2^2$$

- oracle estimator

$$\widehat{\theta}_\Omega = U_\Omega^\dagger y, \quad \widehat{\theta}_{\Omega^c} = 0,$$

where $U = A\Psi$

Outline

Notation

- redundancy coefficient $\gamma = M/K$
- $\mathrm{MSNR} = M\sigma_A^2 p_x/\mathrm{tr}(\Sigma_n)$
- averaged relative power of measurement to perturbation $\eta = \sigma_A^2/\sigma_E^2$

Lower and upper bounds on normalized mean square error

Proposition 1. Assume $\Psi = \mathbf{I}$, $\gamma^2 \gg 1$, and $K \gg 1$. For $t > 0$ and $C, c > 0$ depending only on $\sqrt{M}$ and σ_A, denote $\theta_0 = \left(\frac{\sqrt{M} - C\sqrt{K} - t}{\sqrt{M}}\right)^2$, $f_{M,K,\theta_0} = \frac{1}{\theta_0}\left(1 - \frac{(1-\theta_0)^2}{1-\theta_0+\frac{K+\gamma-1}{\gamma K}}\right)$, and $\gamma_f = \frac{\gamma}{f_{M,K,\theta_0}}$.

For the oracle estimator,

$$\mathrm{mse}_0 \geq \frac{1}{\gamma\eta} + \frac{1}{\gamma}\frac{1}{\mathrm{MSNR}} \tag{1}$$

with probability $1 - p_S$, and

$$\mathrm{mse}_0 \leq \frac{1}{\gamma_f\eta} + \frac{1}{\gamma_f}\frac{1}{\mathrm{MSNR}} \tag{2}$$

with probability exceeding $1 - 1/\gamma^2 - 2\exp(-ct^2) - p_S$, where

$$p_S = (1 + o(1))K(K-1)/2^M = \mathcal{O}(K^2/2^M). \tag{3}$$

Lower and upper bounds on normalized mean square error

Corollary 1. When $\gamma \gg 1$, let $\widehat{\gamma} = \gamma^2/(\gamma+1)$. We have $\gamma_f \geq \widehat{\gamma}$. Thus, we can introduce the following lower and upper bounds

$$\frac{1}{\gamma\eta} + \frac{1}{\gamma}\frac{1}{\mathrm{MSNR}} \leq \mathrm{mse}_0 \leq \frac{1}{\widehat{\gamma}\eta} + \frac{1}{\widehat{\gamma}}\frac{1}{\mathrm{MSNR}},$$

which hold with high probability.

- When $E = 0$, we have $1/\eta = 0$, and

$$\frac{1}{\gamma}\frac{1}{\mathrm{MSNR}} \leq \mathrm{mse}_0 \leq \frac{1}{\widehat{\gamma}}\frac{1}{\mathrm{MSNR}}$$

- When $n = 0$, we have $\mathrm{MSNR} = \infty$, and

$$\frac{1}{\gamma\eta} \leq \mathrm{mse}_0 \leq \frac{1}{\widehat{\gamma}\eta}$$

Outline

Sketch of the proof

- $\widehat{\theta}_\Omega = A_\Omega^\dagger y$, $\widehat{\theta}_{\Omega^c} = 0$
- given that columns of A_Ω are independent

$$\mathrm{mse} = \mathbb{E}\|\theta_\Omega - \widehat{\theta}_\Omega\|_2^2 = \mathbb{E}\|A_\Omega^\dagger z\|_2^2 = \mathbb{E}[z^\mathsf{T} B^\dagger z] = \sum_{i=1}^{K} \frac{\mathbb{E}(w_i^2)}{\lambda_i}$$

- $z = E\theta + n$, $B = A_\Omega A_\Omega^\mathsf{T}$, $w = Qz$
- $\mathbb{E}(B) = Q^T \Lambda Q$ is the eigen-decomposition
- $\mathbb{E}(w_i^2) = \sigma_n^2 + \sigma_E^2 \mathbb{E}(x^\mathsf{T} x)$

Sketch of the proof

- Columns of A_Ω are linearly dependent with probability p_S defined in (3).
- bounds on eigenvalues of $B^\dagger$
  - ***Lemma 2.*** If the columns of A_Ω are independent,

    $$\mathrm{tr}(B^\dagger) \geq \sigma_A^{-2} K/M, \quad \text{a.s.}.$$

  - ***Lemma 3.*** If the columns of A_Ω are independent,

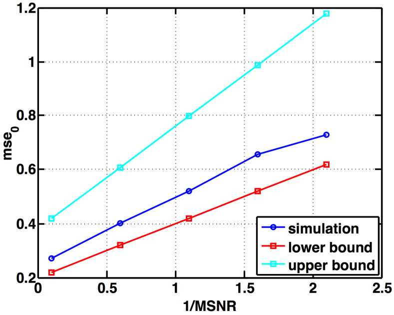
    $$\mathrm{tr}(B^\dagger) \leq \frac{K}{M\sigma_A^2} f_{M,K,\theta_0}$$

    holds with probability at least $1 - 1/\gamma^2 - \exp(-ct^2)$, where θ_0, f_{M,K,θ_0}, C, and $c > 0$ are defined in Proposition 1.

Outline

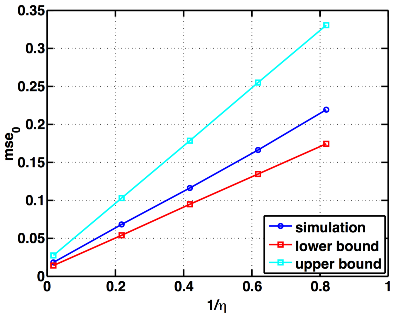
Lower and upper bounds of mse_0 versus MSNR

- $N = 625$, $K = 20$, $t = 1$, $\sigma_A = 0.1$, and $C = 1$
- Components of θ_Ω: i.i.d. standard normal distribution
- averaged from 200 trials



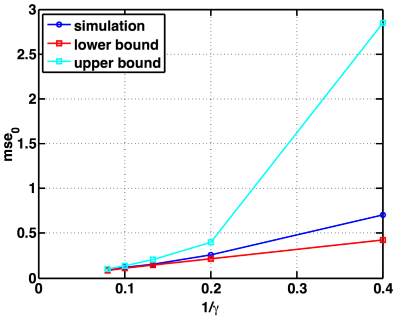
- $M = 100$, $\sigma_E = 0.1$, and $1/\mathrm{MSNR} \in \{0.1, 0.6, 1.1, 1.6, 2.1\}$

Lower and upper bounds of mse_0 versus η



- $M = 100$, $\sigma_n = 0.1$, and $1/\eta \in \{0.02, 0.22, 0.42, 0.62, 0.82\}$

Lower and upper bounds of mse_0 versus γ



- $\sigma_n = \sigma_E = 0.1$ and $\gamma \in \{2.5, 5, 7.5, 10, 12.5\}$

Outline

Conclusion

- performance of the oracle receiver in sparse signal recovery with Bernoulli sensing matrix
- measurement noise and sensing matrix perturbation
- lower and upper bounds of the normalized mean square error
- numerical results verify the bounds