Graph-based Reconstruction of Time-varying Spatial Signals

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1 Preliminaries

2 Problem Formulation

3 Problem Solver

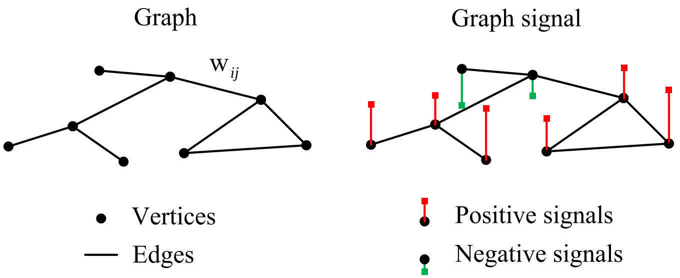
4 Experiments

Graph and Graph Signal

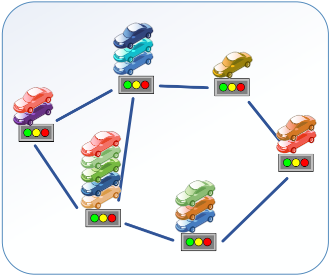
Undirected, connected and weighted graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathbf{W})$

- $\mathcal{V}$ is the set of vertices
- $\mathcal{E}$ is the set of edges
- $\mathbf{W}$ is the weighted adjacency matrix

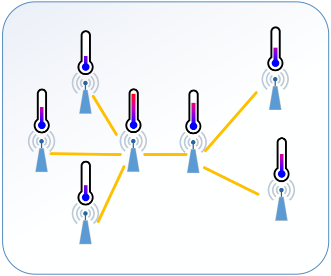
Graph signal $\mathbf{f} \in \mathbb{R}^N$, or a mapping $f : \mathcal{V} \to \mathbb{R}$



Examples of Graph Signals



Traffic queue length
on road network



Temperatures
on sensor network

Figure: Examples of graph signals.

Graph signal sampling and reconstruction

- Sampling set $\mathcal{S} \subseteq \mathcal{V}$
- Reconstruct $\mathbf{f}$ from the known samples $\{f(u)\}_{u \in \mathcal{S}}$
- Conditions: Bandlimited or smooth signals on graph

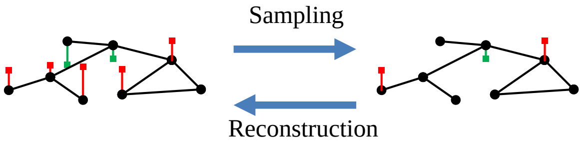


Figure: Sampling and reconstruction of graph signals.

The frequency domain of graph signals

- Laplacian $\mathbf{L} = \mathbf{D} - \mathbf{W}$
- Fourier basis $\{\mathbf{u}_k\}_{1 \le k \le N}$
- Frequencies $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_N$
- $\hat{f}(\lambda_k) = \langle \mathbf{f}, \mathbf{u}_k \rangle$

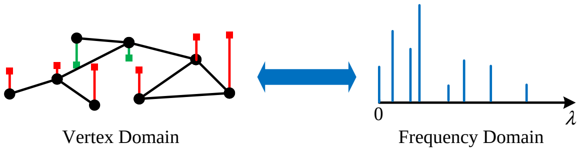
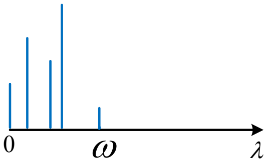


Figure: Vertex and frequency domains of a graph signal.

Bandlimited Graph Signal

Bandlimited graph signal

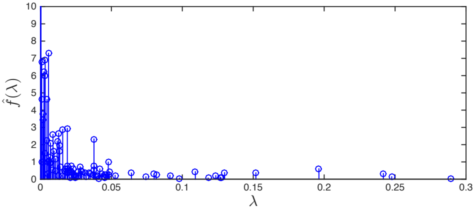
- The subspace of ω-bandlimited signals is called Paley-Wiener space $PW_\omega(\mathcal{G}) \triangleq \text{span}\{\mathbf{u}_i | \lambda_i \leq \omega\}$
- Bandlimited graph signal $\mathbf{f} \in PW_\omega(\mathcal{G})$
- A graph signal in Paley-Wiener space can be uniquely determined by its uniqueness set.



Reconstruction of Smooth Graph Signal

Smooth graph signal

- Not excatly bandlimited
- Existing high frequency components



Optimization problem

$$\min_{\mathbf{x}} \frac{1}{2}||\mathbf{J}\mathbf{x} - \mathbf{y}||_2^2 + \frac{\lambda}{2}||\mathbf{H}\mathbf{x}||_2^2. \quad (1)$$

- $\mathbf{J}$: sampling operator
- $\mathbf{y}$: sampled data
- $\mathbf{H}$: high-pass graph filter

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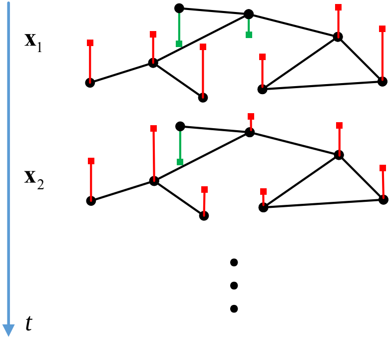
Time-varying Graph Signal

- $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_T]$

If signals $\mathbf{x}_i$ are smooth on the graph:



$$\min_{\mathbf{x}_1} \frac{1}{2} \| \mathbf{J}_1 \mathbf{x}_1 - \mathbf{y}_1 \|_2^2 + \frac{\lambda}{2} \mathbf{x}_1^{\mathrm{T}} \mathbf{L} \mathbf{x}_1$$

$$\min_{\mathbf{x}_2} \frac{1}{2} \| \mathbf{J}_2 \mathbf{x}_2 - \mathbf{y}_2 \|_2^2 + \frac{\lambda}{2} \mathbf{x}_2^{\mathrm{T}} \mathbf{L} \mathbf{x}_2$$

$$\vdots$$

Smoothness of Time-varying Graph Signal

- For most real-world datasets, they are not smooth enough on graph
- The temporal difference signal often demonstrates better smoothness on graph than the original signal

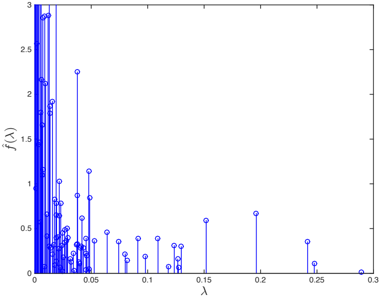


Figure: Frequency domain representation of $\mathbf{x}_1$.

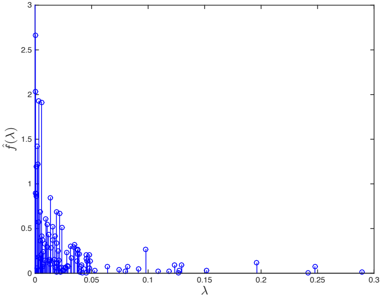


Figure: Frequency domain representation of $\mathbf{x}_2 - \mathbf{x}_1$.

Problem Formulation

Optimization problem

$$\min_{\mathbf{x}} \frac{1}{2}||\mathbf{J}\mathbf{x} - \mathbf{y}||_2^2 + \frac{\lambda}{2}(\mathbf{T}\mathbf{x})^{\mathsf{T}}\mathcal{L}\mathbf{T}\mathbf{x}, \tag{2}$$

- $\mathbf{x} = \text{vec}(\mathbf{X})$, where $\mathbf{X} = [\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_T]$
- $\mathbf{J} = \text{diag}\{\mathbf{J}_1, \mathbf{J}_2, \cdots, \mathbf{J}_T\}$
- $\mathbf{y}$ is sampled data
- $\mathcal{L} = \text{diag}\{\mathbf{L}, \mathbf{L}, \cdots, \mathbf{L}\}$
- $\mathbf{T}$ is the differential operator of $\mathbf{x}$ along the time direction

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Reconstruction Algorithm of The Proposed Model

Input: Sampled data $\mathbf{y}$, sampling operator $\mathbf{J}$, Laplacian matrix $\mathbf{L}$, regularization parameter λ, line search parameters α, β;

Output: Reconstructed signal $\mathbf{x}_k$;

Initialization: $\mathbf{x}_0 = 0$; $\mathbf{g}_0 = \nabla f(\mathbf{x}_0)$; $\Delta\mathbf{x}_0 = -\mathbf{g}_0$;

Loop:

1. Backtracking line-search.

   $t = 1$;

   while $f(\mathbf{x}_k + t\Delta\mathbf{x}_k) > f(\mathbf{x}_k) + \alpha t \mathbf{g}_k^{\mathsf{H}} \Delta\mathbf{x}_k$

   $\quad t = \beta t$;

   $\mathbf{x}_{k+1} = \mathbf{x}_k + t\Delta\mathbf{x}_k$;

2. Update search direction.

   $\mathbf{g}_{k+1} = \nabla f(\mathbf{x}_{k+1})$;

   $\gamma = \frac{||\mathbf{g}_{k+1}||_2^2}{||\mathbf{g}_k||_2^2}$;

   $\Delta\mathbf{x}_{k+1} = -\mathbf{g}_{k+1} + \gamma\Delta\mathbf{x}_k$;

Until: The stop condition is satisfied.

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The sea surface temperature dataset

Source data

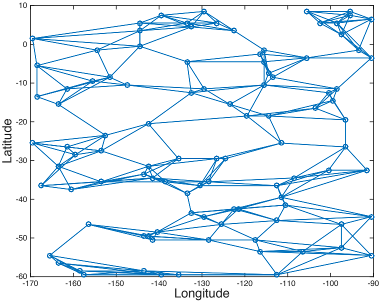
- temporal resolution : monthly from 1870 to 2014
- spatial resolution : 1° latitude $\times 1^\circ$ longitude global grid

Simulation data

- randomly selecting 100 points on the Pacific
- time length of 600 months

Graph construction

- established by k-nearest neighbor method
- the edge weights are inversely proportional to the square of their distance



Sampling rate : 40%

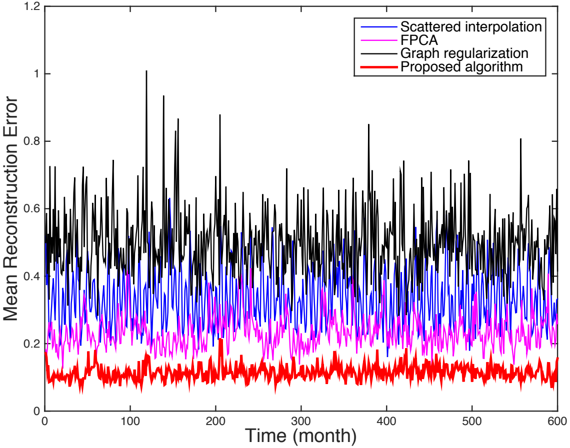


Figure: The mean reconstruction error versus time.

Varying sampling rate

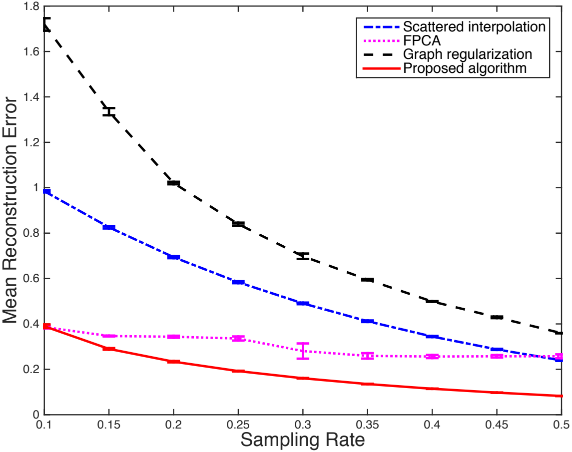


Figure: The mean reconstruction error under different sampling rate.

The sensor network dataset

Source data

- spatial resolution : 53 sensors distributed around the laboratory
- temporal resolution : every 30 seconds from February 28th, 2004
- the section between 01:06 and 17:56 on February 28th, 2004 is selected

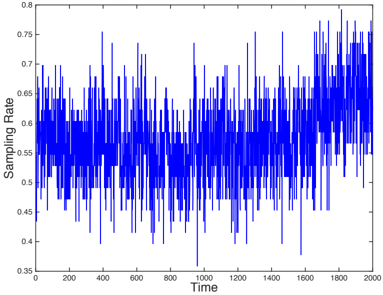


Figure: Sampling rate of the sensor network data.

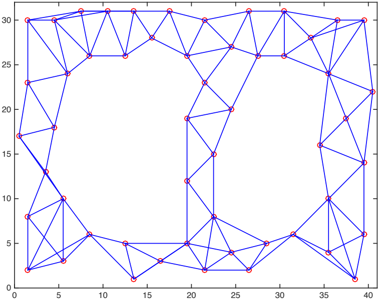


Figure: Graph established by 4-nearest neighbors.

Experimental Results

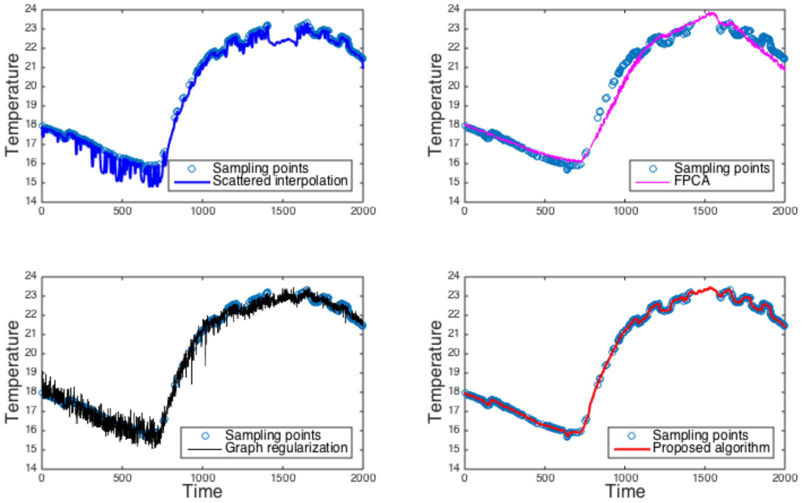


Figure: The reconstructed temperature of one sensor.

References

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- Least-square reconstruction (S. Narang, A. Gadde and A. Ortega, 2013)
- Iterative least square reconstruction (S. Narang, A. Gadde, E. Sanou and A. Ortega, 2013)
- TV-minimization reconstruction (S. Chen, A. Sandryhaila, J. Moura and J. Kovacevic, 2014)
- Local-set-based reconstruction (X. Wang, P. Liu and Y. Gu, 2014)
- Sampling theorem (A. Anis, A. Gadde and A. Ortega, 2014; S. Chen, A. Sandryhaila, J. Moura and J. Kovacevic, 2015)
- Distributed algorithms (X. Wang, M. Wang and Y. Gu, 2015; S. Chen, A. Sandryhaila, and J. Kovacevic, 2015)
- Reconstruction through percolation (S. Segarra, A. Marques, G. Leus, A. Ribeiro, 2015)
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The End

Thank you!